Published by Institute of Physics Publishing for SISSA

Received: *March 28, 2006*
Accepted: *April 10, 2006*
Published: *April 28, 2006*

Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory

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Abstract: Schnabl recently constructed an analytic solution for tachyon condensation in Witten's open string field theory. The solution consists of two pieces. Only the first piece is involved in proving that the solution satisfies the equation of motion when contracted with any state in the Fock space. On the other hand, both pieces contribute in evaluating the kinetic term to reproduce the value predicted by Sen's conjecture. We therefore need to understand why the second piece is necessary. We evaluate the cubic term of the string field theory action for Schnabl's solution and use it to show that the second piece is necessary for the equation of motion contracted with the solution itself to be satisfied. We also present the solution in various forms including a pure-gauge configuration and provide simpler proofs that it satisfies the equation of motion.

Keywords: Tachyon Condensation, Bosonic Strings, String Field Theory.

Contents

1. Introduction

Witten's cubic open string field theory [1] has been used to calculate the potential of the open string tachyon in bosonic string theory, and convincing evidence has been accumulated for the existence of a nontrivial critical point in the potential by an approximation scheme called level truncation [2 − 8]. The depth of the potential at the critical point coincides with the D25-brane tension with impressive precision [5 − 8], providing strong support for Sen's conjecture [9, 10]. The equation of motion of Witten's string field theory [1] is given by

$$Q_B\Psi + \Psi * \Psi = 0\,, \tag{1.1}$$

where Q_B is the BRST operator, and the product in the second term is Witten's star product. The value of the potential at the critical point predicted by Sen's conjecture is

$$\frac{1}{\alpha'^3 g_T^2}\left[\frac{1}{2}\langle\,\Psi, Q_B\Psi\,\rangle + \frac{1}{3}\langle\,\Psi, \Psi * \Psi\,\rangle\right] = -\,\frac{1}{2\,\pi^2\alpha'^3 g_T^2}\,, \tag{1.2}$$

where g_T is the on-shell three-tachyon coupling constant.[1] We only consider translationally invariant string field configurations in this paper, and the inner product is defined to be the standard BPZ inner product divided by the space-time volume factor. Combining this

[1]See appendix A of [11] for a derivation of the relation between the D25-brane tension and g_T from equations in Polchinski's book [12].

with the equation of motion, the prediction from Sen's conjecture therefore amounts to

$$\langle \Psi, Q_B \Psi \rangle = -\frac{3}{\pi^2}, \tag{1.3}$$

$$\langle \Psi, \Psi * \Psi \rangle = \frac{3}{\pi^2}. \tag{1.4}$$

Schnabl recently constructed an analytic solution for the tachyon vacuum in Witten's string field theory [13]. The solution Ψ consists of two pieces and is defined by a limit:

$$\Psi = \lim_{N\to\infty} \left[\sum_{n=0}^{N} \frac{d}{dn}\psi_n - \psi_N \right] \equiv \lim_{N\to\infty} \left[\sum_{n=0}^{N} \psi'_n - \psi_N \right], \tag{1.5}$$

where the state ψ_n defined for any real n in the range $n \geq 0$ is made of the wedge state [14–16] with some operator insertions.[2] It was shown that the string field Ψ in (1.5) satisfies the equation of motion of Witten's string field theory contracted with any state ϕ in the Fock space:

$$\langle \phi, Q_B \Psi \rangle + \langle \phi, \Psi * \Psi \rangle = 0. \tag{1.6}$$

The kinetic term of the Witten's string field theory action was then evaluated for the solution Ψ, and the value (1.3) predicted by Sen's conjecture was analytically reproduced.

Actually, the ψ_N piece of the solution Ψ in (1.5) does not contribute to inner products with states in the Fock space. Namely,

$$\lim_{N\to\infty} \langle \phi, \psi_N \rangle = 0 \tag{1.7}$$

for any state ϕ in the Fock space. What was really shown in [13] is

$$\sum_{n=0}^{\infty} \langle \phi, Q_B \psi'_n \rangle + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle \phi, \psi'_n * \psi'_m \rangle = 0 \tag{1.8}$$

for any state ϕ in the Fock space. In other words, the ψ_N piece is not required by (1.6). On the other hand, the ψ_N piece does contribute in evaluating the kinetic term of the Witten's string field theory action. More explicitly, the following quantities were calculated in [13]:

$$\mathcal{K}_2 = \lim_{N\to\infty} \sum_{n=0}^{N}\sum_{m=0}^{N} \langle \psi'_n, Q_B \psi'_m \rangle = \frac{1}{2} - \frac{1}{\pi^2}, \tag{1.9}$$

$$\mathcal{K}_1 = \lim_{N\to\infty} \sum_{m=0}^{N} \langle \psi_N, Q_B \psi'_m \rangle = \frac{1}{2} + \frac{2}{\pi^2}, \tag{1.10}$$

$$\mathcal{K}_0 = \lim_{N\to\infty} \langle \psi_N, Q_B \psi_N \rangle = \frac{1}{2} + \frac{2}{\pi^2}. \tag{1.11}$$

The inner product $\langle \Psi, Q_B \Psi \rangle$ for the solution Ψ in (1.5) is then

$$\langle \Psi, Q_B \Psi \rangle = \mathcal{K}_2 - 2\,\mathcal{K}_1 + \mathcal{K}_0 = -\frac{3}{\pi^2}. \tag{1.12}$$

In order to reproduce the value (1.3) predicted by Sen's conjecture, the ψ_N piece is really necessary. In particular, the coefficient in front of ψ_N in (1.5) must be -1.

[2] Our notion is slightly different from Schnabl's so that the solution (1.5) differs from that in [13] by an overall sign. See the beginning of the next section for more details.

In fact, Schnabl first hypothesized the solution in the following form:

$$\Psi = -\sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{d^n}{dm^n} \psi_m \bigg|_{m=0}, \tag{1.13}$$

where B_n's are the Bernoulli numbers. It was then brought to the form (1.5) using the Euler-Maclaurin formula. In this sense, it is natural to add the ψ_N piece with the coefficient -1. However, this is not the requirement that the equation of motion be satisfied. If the ψ_N piece is irrelevant to the equation of motion, solutions of the form

$$\lim_{N\to\infty} \left[\sum_{n=0}^{N} \psi'_n - \alpha\, \psi_N \right] \tag{1.14}$$

with any real α will be equally valid. The evaluation of the kinetic term is well defined for any α, and it yields a value inconsistent with Sen's conjecture unless $\alpha = 1$. Let us state the puzzle as follows.

1. Why do we need the ψ_N piece? What determines the coefficient in front of ψ_N?

There is another question, which turns out to be related to the above puzzle. In evaluating the Witten's string field theory action in [13], it was implicitly assumed that the equation of motion is satisfied when it is contracted with the solution itself:

$$\langle\, \Psi, Q_B \Psi \,\rangle + \langle\, \Psi, \Psi * \Psi \,\rangle = 0\,. \tag{1.15}$$

However, this is notoriously subtle in string field theory because the solution is usually outside the Fock space. For example, the formal exact solution based on the identity state [17, 18] satisfies the equation of motion when contracted with any state in the Fock space, but the calculations of the inner products in (1.15) are not even well defined. Another example is the twisted butterfly state [19–21] in vacuum string field theory [22–24]. It solves the equation of motion when contracted with any state in the Fock space, but it does not satisfy the equation when contracted with the solution itself [25], which indicates that the assumption of the matter-ghost factorization in vacuum string field theory needs to be reconsidered [26]. While a systematic approach to accomplish the compatibility of (1.6) and (1.15) has been developed in [27, 28, 26], it relies on a series expansion and the compatibility is only approximate. It is therefore crucially important whether or not Schnabl's solution satisfies (1.15). Our question is as follows.

2. Does the solution satisfy the equation of motion even when it is contracted with the solution itself?

In order to address this question, it is necessary to evaluate the cubic term of the string field theory action for Schnabl's solution to see if the value (1.4) is reproduced. We evaluate the following quantities in this paper:

$$\mathcal{V}_3 = \lim_{N\to\infty} \sum_{n=0}^{N}\sum_{m=0}^{N}\sum_{k=0}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle = \frac{3}{\pi^2} - \frac{3\sqrt{3}}{2\pi}\,, \tag{1.16}$$

$$\mathcal{V}_2 = \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{k=0}^{N} \langle\, \psi_N, \psi'_m * \psi'_k \,\rangle = -\frac{3\sqrt{3}}{2\pi}\,, \tag{1.17}$$

$$\mathcal{V}_1 = \lim_{N\to\infty} \sum_{n=0}^{N} \langle\, \psi'_n, \psi_N * \psi_N \,\rangle = -\frac{3\sqrt{3}}{2\pi}\,, \tag{1.18}$$

$$\mathcal{V}_0 = \lim_{N\to\infty} \langle\, \psi_N, \psi_N * \psi_N \,\rangle = -\frac{3\sqrt{3}}{2\pi}\,. \tag{1.19}$$

The inner product $\langle\, \Psi, \Psi * \Psi \,\rangle$ for the solution Ψ in (1.5) is then given by

$$\langle\, \Psi, \Psi * \Psi \,\rangle = \mathcal{V}_3 - 3\,\mathcal{V}_2 + 3\,\mathcal{V}_1 - \mathcal{V}_0 = \frac{3}{\pi^2}\,. \tag{1.20}$$

The equation of motion (1.15) is *not* satisfied without the ψ_N piece because $\mathcal{K}_2 + \mathcal{V}_3 \neq 0$, but it *is* nontrivially satisfied when the ψ_N piece is included with the coefficient -1. This is the main result of the paper. We believe that this dispels the questions raised earlier and provides nontrivial evidence for Schnabl's solution.

Schnabl's solution will definitely play an important role in developing vacuum string field theory further or in constructing different analytic solutions of Witten's string field theory. It is not clear at this point, however, what aspects of the solution will be crucial for future development. We therefore present the solution in various forms including a pure-gauge configuration and provide simpler proofs that it satisfies the equation of motion. We hope that this helps understand the solution better.

The organization of the paper is as follows. In section 2, we present the string states ψ_n and ψ'_n in the conformal field theory (CFT) formulation of string field theory [29, 30]. We show that Schnabl's solution satisfies the equation of motion first in the CFT formulation in subsection 3.1, and then we present a purely algebraic proof in subsection 3.2. We also express Schnabl's solution in the half-string picture in subsection 3.3 and as a pure-gauge configuration in subsection 3.4. The evaluation of the kinetic term of the string field theory action for Schnabl's solution in [13] is reproduced in a different way in section 4, and the cubic term of the string field theory action is evaluated for Schnabl's solution in section 5. Section 6 is devoted to conclusions.

2. Wedge state with operator insertions

Let us first explain the difference between Schnabl's notation and ours. Schnabl's *left* is our *right* and Schnabl's *right* is our *left*. When an operator $\mathcal{O}$ is defined by an integral along the unit circle,

$$\mathcal{O} = \oint \frac{d\xi}{2\pi i}\,\varphi(\xi)\,, \tag{2.1}$$

where $\varphi(\xi)$ can be a field or a field multiplied by a function of ξ, we define $\mathcal{O}^R$ and $\mathcal{O}^L$ by

$$\mathcal{O}^R = \int_{C_R} \frac{d\xi}{2\pi i}\,\varphi(\xi)\,, \qquad \mathcal{O}^L = \int_{C_L} \frac{d\xi}{2\pi i}\,\varphi(\xi)\,, \tag{2.2}$$

where the contour C_R runs along the right half of the unit circle from $-i$ to i counterclockwise and the contour C_L runs along the left half of the unit circle from i to $-i$ counterclockwise. Schnabl's $\mathcal{O}^L$ is our $\mathcal{O}^R$, and Schnabl's $\mathcal{O}^R$ is our $\mathcal{O}^L$. In this paper, we use the following operators of these forms:

$$B_1^R = \int_{C_R} \frac{d\xi}{2\pi i}\,(\xi^2+1)\,b(\xi)\,, \qquad B_1^L = \int_{C_L} \frac{d\xi}{2\pi i}\,(\xi^2+1)\,b(\xi)\,, \tag{2.3}$$

$$K_1^R = \int_{C_R} \frac{d\xi}{2\pi i}\,(\xi^2+1)\,T(\xi)\,, \qquad K_1^L = \int_{C_L} \frac{d\xi}{2\pi i}\,(\xi^2+1)\,T(\xi)\,, \tag{2.4}$$

where $b(\xi)$ is the b ghost and $T(\xi)$ is the energy-momentum tensor.

This difference in the definition of $left$ and $right$ also affects the definition of the star product. Schnabl's $A * B$ is our $(-1)^{AB} B * A$.[3] Here and in what follows a string field in the exponent of -1 denotes its Grassmann property: it is 0 mod 2 for a Grassmann-even string field and is 1 mod 2 for a Grassmann-odd string field. Schnabl solved the equation $Q_B \Psi + \Psi * \Psi = 0$, which corresponds to $Q_B \Psi - \Psi * \Psi = 0$ in our notation. We solve $Q_B \Psi + \Psi * \Psi = 0$ in our notation so that the solution in this paper should differ from that in [13] by an overall minus sign.

The string state ψ_n introduced by Schnabl in [13] takes the form of the wedge state with some operator insertions. Let us review the definition of the wedge state [14–16]. The wedge state $|n\rangle$ for any real n in the range $n > 1$ can be defined by its inner products with states in the Fock space. For any state ϕ in the Fock space, the inner product $\langle\, \phi, n \,\rangle$ is given by

$$\langle\, \phi, n \,\rangle = \langle\, f_n \circ \phi(0) \,\rangle_{\rm UHP}\,, \tag{2.5}$$

where $\phi(0)$ is the operator corresponding to the state ϕ in the state-operator mapping. We use the notation $f \circ \mathcal{O}(\xi)$ for the operator mapped from $\mathcal{O}(\xi)$ by a conformal transformation $f(\xi)$. When the operator $\mathcal{O}$ is primary with dimension h, $f \circ \mathcal{O}(\xi)$ is as follows:

$$f \circ \mathcal{O}(\xi) = \left(\frac{df(\xi)}{d\xi} \right)^h \mathcal{O}(f(\xi))\,. \tag{2.6}$$

The conformal transformation $f_n(\xi)$ in (2.5) is given by

$$f_n(\xi) = \frac{n}{2} \tan\left(\frac{2}{n} \arctan \xi \right)\,. \tag{2.7}$$

The correlation function is evaluated on the upper-half plane as indicated by the subscript UHP. As in the definition of the inner product, we divide correlation functions by the overall space-time volume factor. Our normalization of correlation functions is given by

$$\langle\, c(w_1)\, c(w_2)\, c(w_3) \,\rangle_{\rm UHP} = (w_1 - w_2)(w_1 - w_3)(w_2 - w_3)\,, \tag{2.8}$$

where $c(w)$ is the c ghost. We use the doubling trick throughout the paper. The normalization of the state-operator mapping is fixed by the condition that the $SL(2,R)$-invariant vacuum $|0\rangle$ corresponds to the identity operator. The normalization of the inner product is also fixed by this together with (2.8).

[3] In our definition, the *left* string field A in $A * B$ is mapped to the *left* of the string field B in the complex plane for the glued surface in the CFT formulation, and the operators associated with A are located to the *left* of those associated with B in correlation functions used in the CFT formulation. See (3.8), for example. Schnabl adopted the definition used in a series of papers by Rastelli, Sen and Zwiebach such as [15] or in a review [31].

The inner product (2.5) can also be written in terms of a correlation function on a semi-infinite cylinder. We denote the semi-infinite cylinder obtained from the upper-half plane of z with the identification $z + \ell \simeq z$ by C_ℓ. The inner product $\langle\, \phi, n \,\rangle$ in terms of a correlation function on $C_{\frac{\pi n}{2}}$ is given by

$$\langle\, \phi, n \,\rangle = \langle\, f_\infty \circ \phi(0) \,\rangle_{C_{\frac{\pi n}{2}}} \,, \tag{2.9}$$

where

$$f_\infty(\xi) = \arctan \xi \,. \tag{2.10}$$

We in fact mostly use this expression of $\langle\, \phi, n \,\rangle$ in this paper. The expression (2.5) can be derived from this in the following way. First map $C_{\frac{\pi n}{2}}$ to C_π by the dilatation $z' = \frac{2}{n}\, z$, where $z = \arctan \xi$ is the coordinate on $C_{\frac{\pi n}{2}}$. Then the conformal transformation $z'' = \tan z'$ maps C_π to the upper-half plane. We can further perform the dilatation $w = \frac{n}{2}\, z''$ which maps the upper-half plane to itself so that the combined transformation $w = f_n(\xi)$ has a limit as $n \to \infty$. After these three transformations, the inner product $\langle\, \phi, n \,\rangle$ is given by (2.5) in the w coordinate.

The z coordinate given by $z = \arctan \xi$ is very useful in dealing with the star product [14, 15, 13]. The upper half of the unit disk is mapped to a semi-infinite strip with a width of $\pi/2$ by the conformal transformation $z = \arctan \xi$, and the left and right halves of the open string are mapped to semi-infinite lines parallel to the imaginary axis. The right half of one string and the left half of the other string are glued together in Witten's star product so that star products of states in the Fock space can be obtained simply by translation in the z coordinate. Inner products of states in the Fock space are also simple in the z coordinate, and they are obtained by first taking the star product of the two states and then by gluing together the left and right halves of the resulting string state. The vacuum state $|0\rangle$ corresponds to a semi-infinite strip with a width of $\pi/2$, and no operators are inserted. The state ϕ in the Fock space corresponds to a semi-infinite strip with a width of $\pi/2$ which has an insertion of $f_\infty \circ \phi(0)$ at the origin. If we subtract the piece coming from the state ϕ in the expression of $\langle\, \phi, n \,\rangle$ in (2.9), the remaining surface is a semi-infinite strip with a width of $\pi(n-1)/2$, and there are no operator insertions. When n is an integer, the wedge state $|n\rangle$ therefore coincides with a product of vacuum states:

$$|n\rangle = \underbrace{|0\rangle * |0\rangle * \ldots * |0\rangle}_{n-1} \,. \tag{2.11}$$

The string field ψ_n defined for any real n in the range $n \geq 0$ is made of the wedge state $|n+2\rangle$ with some operator insertions. It can also be defined by its inner products with states in the Fock space, and those inner products can be expressed by correlation functions on the semi-infinite cylinder $C_{\pi(n+2)/2}$. Let us start with the following expression of ψ_n for $n > 1$ given in footnote 16 of [13]:

$$\psi_n = \frac{1}{\pi}\, c_1\, |0\rangle * (B_1^L - B_1^R)\, |n\rangle * c_1\, |0\rangle \,. \tag{2.12}$$

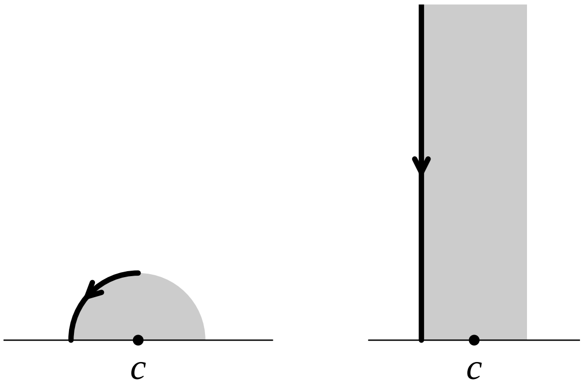

Figure 1: Representations of the state $B_1^L c_1 |0\rangle$. The left figure is the representation in the local coordinate ξ. The state $c_1 |0\rangle$ corresponds to the operator $c(0)$, and the thick line represents the contour of the integral for B_1^L defined in (2.3). The contour runs from i to -1 along the unit circle counterclockwise before using the doubling trick. The right figure is the representation after the conformal transformation $z = f_\infty(\xi) = \arctan \xi$. The c ghost at the origin remains the same because $f_\infty \circ c(0) = c(0)$. The operator B_1^L is mapped to B defined in (2.14), and the contour of the integral represented by the thick line is a semi-infinite line before using the doubling trick.

This takes the same form both in Schnabl's notation and in ours.[4] The mode expansion of the c ghost is given by

$$c_n = \oint \frac{d\xi}{2\pi i}\, \xi^{n-2}\, c(\xi)\,, \tag{2.13}$$

where the contour encircles the origin counterclockwise. The state $c_1 |0\rangle$ corresponds to the operator $c(0)$ in the state-operator mapping. Since $f_\infty \circ c(0) = c(0)$, the state $c_1 |0\rangle$ is represented by a semi-infinite strip with a width of $\pi/2$ which has an insertion of $c(0)$ at the midpoint of the finite edge of the strip. The operator B_1^L defined in (2.3) is mapped to

$$B = \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i}\, b(z) \tag{2.14}$$

by the conformal transformation $z = f_\infty(\xi) = \arctan \xi$. Note that the function $\xi^2 + 1$ in (2.3) is precisely canceled by the conformal factor. The operator B is therefore invariant under translation $z \to z + a$ for any real a. The state $B_1^L |\phi\rangle$ for any state $|\phi\rangle$ in the Fock space is represented by a semi-infinite strip with a width of $\pi/2$, where the operators $B\, f_\infty \circ \phi(0)$ are inserted. The contour of the integral in B must be located to the left of $f_\infty \circ \phi(0)$. See figure 1. Similarly, the state $B_1^R |\phi\rangle$ for any state $|\phi\rangle$ in the Fock space is represented by a semi-infinite strip with a width of $\pi/2$, where the operators $-(-1)^\phi f_\infty \circ \phi(0)\, B$ are inserted. Note that the first minus sign comes from reversing the contour of the integral in B_1^R. The contour of the integral in B must be located to the right of $f_\infty \circ \phi(0)$ in this case. It is easy to see from this representation that $B_1^L |0\rangle = -B_1^R |0\rangle$ because no operators are inserted for the vacuum state $|0\rangle$ so that the contour of the

[4] Schnabl's $B_1^L - B_1^R$ corresponds to our $B_1^R - B_1^L$, but the relative minus sign is compensated when the ordering of the star product is reversed.

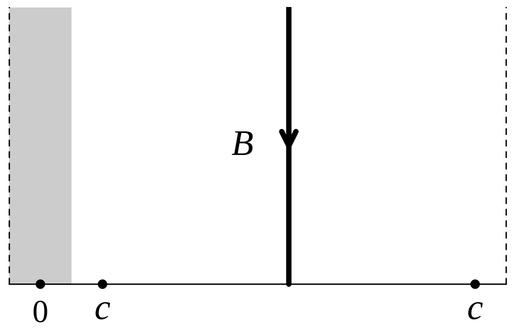

Figure 2: A representation of the inner product $\langle \phi, \psi_n \rangle$. (The overall factor $2/\pi$ is ignored.) The two dashed lines are identified. The shaded region corresponds to the state ϕ, and the operator $f_\infty \circ \phi(0)$ is inserted at the origin. The string state ψ_n is represented by the rest of the region. It is made of the wedge state $|n+2\rangle$ with two c-ghost insertions and one insertion of B defined in (2.14). The contour of the integral before using the doubling trick is represented by the thick line.

integral in B can be freely deformed inside the semi-infinite strip. This generalizes to the wedge state, and the relation $B_1^L |n\rangle = -B_1^R |n\rangle$ holds for any n. Therefore, the factor $(B_1^L - B_1^R)|n\rangle$ in (2.12) is equal to $2\, B_1^L |n\rangle$. We are now ready to express the state ψ_n in the CFT formulation. For any state ϕ in the Fock space, the inner product $\langle \phi, \psi_n \rangle$ is given by

$$\langle \phi, \psi_n \rangle = \frac{2}{\pi} \left\langle f_\infty \circ \phi(0)\, c\left(\frac{\pi}{2}\right)\, B\, c\left(\frac{\pi(n+1)}{2}\right) \right\rangle_{C_{\frac{\pi(n+2)}{2}}} . \tag{2.15}$$

The operator B is defined in (2.14), and if B is located between two operators, the contour of the integral must run between the two operators. In the case of (2.15), the contour must run between the two points $\pi/2$ and $\pi(n+1)/2$. See figure 2. The semi-infinite cylinder $C_{\pi(n+2)/2}$ can be represented by the region $-\pi/4 \le \mathrm{Re}\, z \le \pi(2\,n+3)/4$ of the upper-half complex z plane with the semi-infinite lines at both ends identified, where $\mathrm{Re}\, z$ is the real part of z. The region $-\pi/4 \le \mathrm{Re}\, z \le \pi/4$ with the insertion of $f_\infty \circ \phi(0)$ corresponds to the state ϕ, the region $\pi/4 \le \mathrm{Re}\, z \le 3\,\pi/4$ with a c-ghost insertion corresponds to $c_1 |0\rangle$, the region $3\,\pi/4 \le \mathrm{Re}\, z \le \pi(2\,n+1)/4$ with the insertion of B corresponds to $B_1^L |n\rangle$, and the region $\pi(2\,n+1)/4 \le \mathrm{Re}\, z \le \pi(2\,n+3)/4$ with a c-ghost insertion corresponds to $c_1 |0\rangle$. Since the contour of the integral in B can be deformed as long as it passes between $\pi/2$ and $\pi(n+1)/2$, the string field ψ_n with $n > 1$ can also be written as

$$\psi_n = \frac{2}{\pi}\, c_1 |0\rangle * |n\rangle * B_1^L c_1 |0\rangle\ . \tag{2.16}$$

The wedge state $|n\rangle$ becomes singular when $n < 1$, but the expression (2.15) is well defined in the range $n > 0$. We define ψ_n for $n > 0$ by (2.15), and our definition coincides with that in [13]. The string field ψ_1 is given by

$$\psi_1 = \frac{2}{\pi}\, c_1 |0\rangle * B_1^L c_1 |0\rangle\ . \tag{2.17}$$

The string field ψ_0 can be defined by a limit:

$$\psi_0 \equiv \lim_{n\to 0} \psi_n\, . \tag{2.18}$$

Let us calculate ψ_0 explicitly. The anticommutation relation of B and $c(z)$ is given by

$$\{ B \, , \, c(z) \} = 1 \, . \tag{2.19}$$

Since

$$\lim_{\epsilon \to 0} c(z) \, B \, c(z+\epsilon) = c(z) - \lim_{\epsilon \to 0} c(z) \, c(z+\epsilon) \, B = c(z) \, , \tag{2.20}$$

the inner product $\langle \, \phi, \psi_n \, \rangle$ in the limit $n \to 0$ is given by

$$\lim_{n \to 0} \langle \, \phi, \psi_n \, \rangle = \frac{2}{\pi} \, \Big\langle \, f_\infty \circ \phi(0) \, c \left(\frac{\pi}{2} \right) \Big\rangle_{C_\pi} \, . \tag{2.21}$$

We therefore obtain

$$\psi_0 = \frac{2}{\pi} \, c_1 \, |0\rangle \, . \tag{2.22}$$

Let us next consider the derivative of ψ_n with respect to n. Since the wedge state can be written as [13, 16]

$$|n\rangle = e^{\frac{\pi(n-2)}{2} K_1^L} \, |0\rangle \, , \tag{2.23}$$

the derivative of $|n\rangle$ with respect of n is given by

$$\frac{d}{dn} \, |n\rangle = \frac{\pi}{2} \, K_1^L \, |n\rangle \, . \tag{2.24}$$

The derivative of ψ_n is then given by

$$\psi_n' \equiv \frac{d}{dn} \psi_n = c_1 \, |0\rangle * K_1^L \, |n\rangle * B_1^L c_1 \, |0\rangle \tag{2.25}$$

for $n > 1$. Just as the operator $e^{-\tau L_0}$ creates a piece of the open string world-sheet with a length of τ in the strip coordinate, the operator $e^{\ell K_1^L}$ creates a semi-infinite strip with a width of ℓ. The correspondence between the derivative with respect to n and an insertion of K_1^L multiplied by $\pi/2$ is therefore valid in the whole range $n > 0$. The operator K_1^L defined in (2.4) is mapped to

$$K = \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} \, T(z) \tag{2.26}$$

by the conformal transformation $z = f_\infty(\xi) = \arctan \xi$. Note that K and B commute. The inner product of ψ_n' with any state ϕ in the Fock space is given by

$$\langle \, \phi, \psi_n' \, \rangle = \Big\langle \, f_\infty \circ \phi(0) \, c \left(\frac{\pi}{2} \right) \, B \, K \, c \left(\frac{\pi(n+1)}{2} \right) \Big\rangle_{C_{\frac{\pi(n+2)}{2}}} \tag{2.27}$$

for $n > 0$. See figure 3. As in the case of B, if K is located between two operators, the contour of the integral must run between the two operators. In the case of (2.27), the contour must pass between the two points $\pi/2$ and $\pi(n+1)/2$. Recall that K and B commute. The state $K_1^L \, |\phi\rangle$ for any state $|\phi\rangle$ in the Fock space is represented by a semi-infinite strip with a width of $\pi/2$, where the operators $K \, f_\infty \circ \phi(0)$ are inserted. The contour of the integral in K must be located to the left of $f_\infty \circ \phi(0)$. Similarly, the state $K_1^R \, |\phi\rangle$ for any state $|\phi\rangle$ in the Fock space is represented by a semi-infinite strip with a

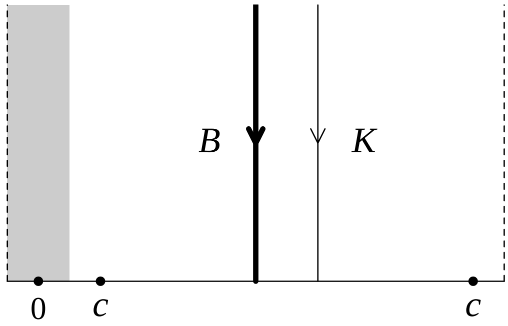

Figure 3: A representation of the inner product $\langle \phi, \psi'_n \rangle$. The two dashed lines are identified. The shaded region corresponds to the state ϕ, and the operator $f_\infty \circ \phi(0)$ is inserted at the origin. The string state ψ'_n is represented by the rest of the region. It is made of ψ_n with an additional insertion of K (2.26) multiplied by $\pi/2$. The contour of the integral for K before using the doubling trick is represented by the thin line.

width of $\pi/2$, where the operators $-f_\infty \circ \phi(0)\, K$ are inserted. Note that the minus sign comes from reversing the contour of the integral in K_1^R. The contour of the integral in K must be located to the right of $f_\infty \circ \phi(0)$ in this case.

The string field ψ'_n can also be written as

$$\psi'_n = c_1 \,|0\rangle * |n\rangle * B_1^L K_1^L c_1 \,|0\rangle \tag{2.28}$$

for $n > 1$, and ψ'_1 is

$$\psi'_1 = c_1 \,|0\rangle * B_1^L K_1^L c_1 \,|0\rangle \;. \tag{2.29}$$

As in the case of ψ_0, the string field ψ'_0 can be defined by a limit:

$$\psi'_0 = \lim_{n\to 0} \psi'_n \,. \tag{2.30}$$

Since

$$\lim_{\epsilon\to 0} c(z)\, K\, c(z+\epsilon) = \lim_{\epsilon\to 0} \left[\, c(z)\, \partial c(z+\epsilon) + c(z)\, c(z+\epsilon)\, K \,\right] = c\partial c(z)\,, \tag{2.31}$$

the product of operators $c(z)\, B\, K\, c(z+\epsilon)$ in the limit $\epsilon \to 0$ can be written as

$$\lim_{\epsilon\to 0} c(z)\, B\, K\, c(z+\epsilon) = \lim_{\epsilon\to 0} K\, c(z+\epsilon) - \lim_{\epsilon\to 0} B\, c(z)\, K\, c(z+\epsilon) = K\, c(z) - B\, c\partial c(z) \tag{2.32}$$

or as

$$\lim_{\epsilon\to 0} c(z)\, B\, K\, c(z+\epsilon) = c(z)\, K - \lim_{\epsilon\to 0} c(z)\, K\, c(z+\epsilon)\, B = c(z)\, K - c\partial c(z)\, B\,. \tag{2.33}$$

Using these formulas, the inner product (2.27) in the limit $n \to 0$ is given by

$$\langle \phi, \psi'_0 \rangle = \lim_{n\to 0} \langle \phi, \psi'_n \rangle = \Big\langle\, f_\infty \circ \phi(0)\, K\, c\left(\frac{\pi}{2}\right) \Big\rangle_{C_\pi} - \Big\langle\, f_\infty \circ \phi(0)\, B\, c\partial c\left(\frac{\pi}{2}\right) \Big\rangle_{C_\pi} \tag{2.34}$$

or by

$$\langle \phi, \psi'_0 \rangle = \lim_{n\to 0} \langle \phi, \psi'_n \rangle = \Big\langle\, f_\infty \circ \phi(0)\, c\left(\frac{\pi}{2}\right)\, K \Big\rangle_{C_\pi} - \Big\langle\, f_\infty \circ \phi(0)\, c\partial c\left(\frac{\pi}{2}\right)\, B \Big\rangle_{C_\pi} \,. \tag{2.35}$$

The string field ψ'_0 is therefore given by

$$\psi'_0 = K_1^L c_1 |0\rangle + B_1^L c_0 c_1 |0\rangle \tag{2.36}$$

or by

$$\psi'_0 = -K_1^R c_1 |0\rangle - B_1^R c_0 c_1 |0\rangle \ . \tag{2.37}$$

Note that the operator $c\partial c(0)$ corresponds to the state $-c_0 c_1 |0\rangle$.

3. Equation of motion

The BRST transformation of the string field ψ'_n with integer n is given by

$$Q_B \psi'_0 = 0 \,, \tag{3.1}$$

$$Q_B \psi'_{n+1} = - \sum_{m=0}^{n} \psi'_m * \psi'_{n-m} \tag{3.2}$$

for $n \geq 0$. Because of this remarkable property, the string field Ψ_λ given by

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} \, \psi'_n \tag{3.3}$$

for any real λ formally satisfies the equation of motion of Witten's string field theory:

$$Q_B \Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0 \,. \tag{3.4}$$

The first piece of Schnabl's solution (1.5) is Ψ_λ with $\lambda = 1$. Recall that the second piece of (1.5) was not involved in proving that (1.5) satisfies the equation of motion contracted with any state in the Fock space in [13]. Solutions with other values of λ were referred to as pure-gauge solutions in [13]. It is not clear for what range of λ the string field Ψ_λ is well defined. We derive (3.1) and (3.2) in various ways in this section.

3.1 Solution in the CFT formulation

Let us first prove (3.1) and (3.2) in the CFT formulation. The BRST transformations of the operators $c(z)$ and B are

$$Q_B \cdot c(z) \equiv \oint \frac{dw}{2\pi i} \, j_B(w) \, c(z) = c\partial c(z) \,, \qquad Q_B \cdot B = K \,, \tag{3.5}$$

where $j_B(w)$ is the BRST current, and the contour of the integral for $Q_B \cdot c(z)$ encircles z counterclockwise. The BRST transformations of the operators $c\partial c(z)$ and K vanish because $Q_B^2 = 0$. It is then easy to show (3.1) contracted with any state ϕ in the Fock space,

$$\langle \, \phi, Q_B \psi'_0 \, \rangle = 0 \,, \tag{3.6}$$

using the expression (2.34) or (2.35).

The inner product $\langle\,\phi, Q_B\psi'_n\,\rangle$ with $n>0$ can be derived from (2.27) as follows:

$$\begin{aligned}\langle\,\phi, Q_B\psi'_n\,\rangle = &\left\langle\, f_\infty\circ\phi(0)\, c\partial c\left(\frac{\pi}{2}\right)\, B\, K\, c\left(\frac{\pi\,(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}}\\ &-\left\langle\, f_\infty\circ\phi(0)\, c\left(\frac{\pi}{2}\right)\, K^2\, c\left(\frac{\pi\,(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}}\\ &+\left\langle\, f_\infty\circ\phi(0)\, c\left(\frac{\pi}{2}\right)\, B\, K\, c\partial c\left(\frac{\pi\,(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}}\,. \end{aligned}\tag{3.7}$$

The inner product for the star product $\langle\,\phi, \psi'_m * \psi'_{n-m}\,\rangle$ with $m, n-m>0$ is given by

$$\begin{aligned}&\langle\,\phi, \psi'_m * \psi'_{n-m}\,\rangle\\ &=\left\langle f_\infty\circ\phi(0)\, c\left(\frac{\pi}{2}\right)KBc\left(\frac{\pi(m+1)}{2}\right)c\left(\frac{\pi(m+2)}{2}\right)BKc\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}.\end{aligned}\tag{3.8}$$

The first operator $f_\infty\circ\phi(0)$ is from the string field ϕ, the next four operators are from ψ'_m, and the last four operators are from ψ'_{n-m}. Using (2.19) and $B^2=0$, we find

$$B\,c(z_1)\,c(z_2)\,B = B\,c(z_1) - B\,c(z_1)\,B\,c(z_2) = B\,c(z_1) - B\,c(z_2)\tag{3.9}$$

for $z_1<z_2$. Then the inner product (3.8) can be written as

$$\begin{aligned}&\langle\,\phi, \psi'_m * \psi'_{n-m}\,\rangle\\ &=\left\langle\, f_\infty\circ\phi(0)\, c\left(\frac{\pi}{2}\right)\, K\,B\,c\left(\frac{\pi(m+1)}{2}\right)\, K\,c\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}\\ &\quad-\left\langle\, f_\infty\circ\phi(0)\, c\left(\frac{\pi}{2}\right)\, K\,B\,c\left(\frac{\pi(m+2)}{2}\right)\, K\,c\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}\,.\end{aligned}\tag{3.10}$$

Note the simple dependence on m. We therefore find

$$\begin{aligned}&\sum_{m=0}^{n}\langle\,\phi, \psi'_m * \psi'_{n-m}\,\rangle\\ &=\lim_{m\to 0}\left\langle\, f_\infty\circ\phi(0)\, c\left(\frac{\pi}{2}\right)\, K\,B\,c\left(\frac{\pi(m+1)}{2}\right)\, K\,c\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}\\ &\quad-\lim_{m\to n}\left\langle\, f_\infty\circ\phi(0)\, c\left(\frac{\pi}{2}\right)\, K\,B\,c\left(\frac{\pi(m+2)}{2}\right)\, K\,c\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}\,.\end{aligned}\tag{3.11}$$

Using the formulas (2.31) and (2.33), the sum can be written as

$$\sum_{m=0}^{n}\langle\,\phi, \psi'_m * \psi'_{n-m}\,\rangle = -\left\langle\, f_\infty\circ\phi(0)\, c\partial c\left(\frac{\pi}{2}\right)\, B\,K\,c\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}$$

$$+\left\langle\, f_\infty\circ\phi(0)\,c\left(\frac{\pi}{2}\right)\,K^2\,c\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}$$
$$-\left\langle\, f_\infty\circ\phi(0)\,c\left(\frac{\pi}{2}\right)\,B\,K\,c\partial c\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}}\,. \quad (3.12)$$

We have thus shown (3.2) contracted with any state ϕ in the Fock space:

$$\langle\,\phi, Q_B\psi'_{n+1}\,\rangle = -\sum_{m=0}^{n}\langle\,\phi, \psi'_m*\psi'_{n-m}\,\rangle \quad (3.13)$$

for any integer n with $n\geq 0$.

Before concluding the subsection, let us mention a connection between the solution in the form (1.5) and that in terms of the Bernoulli numbers (1.13). Using the formula

$$|n\rangle * |\phi\rangle = e^{\frac{\pi(n-1)}{2}K_1^L}\,|\phi\rangle \quad (3.14)$$

for any string field ϕ [13, 16], ψ'_n with $n>1$ can be written as

$$\begin{aligned}\psi'_n &= c_1\,|0\rangle * |n\rangle * K_1^L B_1^L c_1\,|0\rangle = c_1\,|0\rangle * e^{\frac{\pi(n-1)}{2}K_1^L}K_1^L B_1^L c_1\,|0\rangle\\ &= c_1\,|0\rangle * e^{-\frac{\pi}{4}K_1^L}e^{\frac{\pi n}{2}K_1^L}K_1^L B_1^L e^{-\frac{\pi}{4}K_1^L}c_1\,|0\rangle\\ &= e^{\frac{\pi}{4}K_1^R}c_1\,|0\rangle * e^{\frac{\pi n}{2}K_1^L}K_1^L B_1^L e^{-\frac{\pi}{4}K_1^L}c_1\,|0\rangle\,. \end{aligned} \quad (3.15)$$

The expression in the third line is actually valid not only in the range $n>1$ but also in the whole range $n>0$. The actions of the operators $e^{\frac{\pi}{4}K_1^R}$ and $e^{-\frac{\pi}{4}K_1^L}$ could be singular, but it is not difficult to see from (2.27) that the expression in the third line is regular for the entire range $n>0$, and the limit $n\to 0$ is well defined. The solution Ψ_λ in (3.3) can be formally written as

$$\Psi_\lambda = \sum_{n=0}^{\infty}\lambda^{n+1}\,\psi'_n = \frac{2\,\lambda}{\pi}\,e^{\frac{\pi}{4}K_1^R}c_1\,|0\rangle * \frac{\frac{\pi}{2}K_1^L}{1-\lambda\,e^{\frac{\pi}{2}K_1^L}}\,B_1^L e^{-\frac{\pi}{4}K_1^L}c_1\,|0\rangle\,. \quad (3.16)$$

When $\lambda=1$, Ψ_λ takes the form

$$\Psi_{\lambda=1} = -\frac{2}{\pi}\,e^{\frac{\pi}{4}K_1^R}c_1\,|0\rangle * f_B\left(\frac{\pi}{2}K_1^L\right)B_1^L e^{-\frac{\pi}{4}K_1^L}c_1\,|0\rangle\,, \quad (3.17)$$

where

$$f_B(x) = \frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n\,x^n}{n!} \quad (3.18)$$

is the generating function of the Bernoulli numbers B_n. If we expand $1/(e^x-1)$ in $f_B(x)$ in powers of e^x, the first piece of Schnabl's solution (1.5) is reproduced. If we instead expand the function $f_B(x)$ in powers of x, the solution in terms of the Bernoulli numbers (1.13) is obtained. Recall the correspondence between taking a derivative of ψ_n with respect to n and inserting an operator K_1^L multiplied by $\pi/2$ we mentioned below (2.24).

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3.2 Solution by an algebraic construction

It is also possible to prove (3.1) and (3.2) in a more algebraic way. We present such a proof in this subsection.

In this subsection, we define

$$\psi'_0 = K_1^L c_1 |0\rangle + B_1^L c_0 c_1 |0\rangle \,, \tag{3.19}$$

$$\psi'_n = c_1 |0\rangle * |n\rangle * B_1^L K_1^L c_1 |0\rangle \tag{3.20}$$

for integer n in the range $n \geq 1$, where

$$|n\rangle = \underbrace{|0\rangle * |0\rangle * \ldots * |0\rangle}_{n-1} , \tag{3.21}$$

with the understanding that

$$\psi'_1 = c_1 |0\rangle * B_1^L K_1^L c_1 |0\rangle \,. \tag{3.22}$$

In general, the state $|n\rangle$ with $n = 1$ should be understood as

$$|1\rangle * |\phi\rangle = |\phi\rangle \tag{3.23}$$

for any string field ϕ.

Let us first list the equations we use to prove (3.1) and (3.2):

$$Q_B\, (\phi_1 * \phi_2) = (Q_B\, \phi_1) * \phi_2 + (-1)^{\phi_1} \phi_1 * (Q_B\, \phi_2)\,, \tag{3.24}$$

$$Q_B^2 = 0\,, \tag{3.25}$$

$$Q_B\, |0\rangle = 0\,, \tag{3.26}$$

$$Q_B\, c_1\, |0\rangle = \, - \, c_0 c_1 \, |0\rangle \,, \tag{3.27}$$

$$(B_1^R \phi_1) * \phi_2 = \, - \, (-1)^{\phi_1} \phi_1 * (B_1^L \phi_2)\,, \tag{3.28}$$

$$(B_1^L)^2 = (B_1^R)^2 = 0\,, \tag{3.29}$$

$$(B_1^L + B_1^R)\, |0\rangle = 0\,, \tag{3.30}$$

$$(B_1^L + B_1^R)\, c_1\, |0\rangle = |0\rangle \,, \tag{3.31}$$

$$\{ Q_B, B_1^L \} = K_1^L\,, \tag{3.32}$$

$$\{ Q_B, B_1^R \} = K_1^R\,, \tag{3.33}$$

for any string fields ϕ_1 and ϕ_2. We also use the associativity of the star product. It then follows from these equations that for any string fields ϕ_1 and ϕ_2,

$$(K_1^R\, \phi_1) * \phi_2 = -\phi_1 * (K_1^L\, \phi_2)\,, \tag{3.34}$$

$$[Q_B, K_1^L] = 0\,, \tag{3.35}$$

$$[B_1^L, K_1^L] = 0\,, \tag{3.36}$$

$$(K_1^L + K_1^R)\,|0\rangle = 0\,, \tag{3.37}$$

$$B_1^L\,|0\rangle * |0\rangle = |0\rangle * B_1^L\,|0\rangle\;, \tag{3.38}$$

$$K_1^L\,|0\rangle * |0\rangle = |0\rangle * K_1^L\,|0\rangle\;. \tag{3.39}$$

The string field ψ'_n with $n \geq 1$ can be written as

$$\psi'_n = B_1^R c_1\,|0\rangle * |n\rangle * K_1^L c_1\,|0\rangle \tag{3.40}$$

or as

$$\psi'_n = -K_1^R c_1\,|0\rangle * |n\rangle * B_1^L c_1\,|0\rangle\;. \tag{3.41}$$

The string field ψ'_0 is BRST exact:

$$\psi'_0 = K_1^L c_1\,|0\rangle + B_1^L c_0 c_1\,|0\rangle = \{Q_B, B_1^L\}\,c_1\,|0\rangle - B_1^L Q_B\,|0\rangle = Q_B\,B_1^L c_1\,|0\rangle\;. \tag{3.42}$$

This proves (3.1). We also use the following expression of ψ'_0:

$$\psi'_0 = Q_B\,B_1^L c_1\,|0\rangle = Q_B\,|0\rangle - Q_B\,B_1^R c_1\,|0\rangle = -\,Q_B\,B_1^R c_1\,|0\rangle = -\,K_1^R c_1\,|0\rangle - B_1^R c_0 c_1\,|0\rangle\;. \tag{3.43}$$

The string field $Q_B\,\psi'_n$ with $n \geq 1$ is given by

$$Q_B\,\psi'_n = -c_0 c_1\,|0\rangle * |n\rangle * B_1^L K_1^L c_1\,|0\rangle - c_1\,|0\rangle * |n\rangle * (K_1^L)^2 c_1\,|0\rangle - c_1\,|0\rangle * |n\rangle * B_1^L K_1^L c_0 c_1\,|0\rangle\;. \tag{3.44}$$

The star product $\psi'_m * \psi'_{n-m}$ with $n, n-m \geq 1$ is given by

$$\psi'_m * \psi'_{n-m} = -K_1^R c_1\,|0\rangle * |m\rangle * B_1^L c_1\,|0\rangle * B_1^R c_1\,|0\rangle * |n-m\rangle * K_1^L c_1\,|0\rangle\;. \tag{3.45}$$

Since

$$\begin{aligned} B_1^L c_1\,|0\rangle * B_1^R c_1\,|0\rangle &= B_1^L c_1\,|0\rangle * |0\rangle - B_1^L c_1\,|0\rangle * B_1^L c_1\,|0\rangle \\ &= B_1^L c_1\,|0\rangle * |0\rangle + B_1^R B_1^L c_1\,|0\rangle * c_1\,|0\rangle \\ &= B_1^L c_1\,|0\rangle * |0\rangle + B_1^R\,|0\rangle * c_1\,|0\rangle - (B_1^R)^2 c_1\,|0\rangle * c_1\,|0\rangle \\ &= B_1^L c_1\,|0\rangle * |0\rangle - |0\rangle * B_1^L c_1\,|0\rangle\;, \end{aligned} \tag{3.46}$$

the star product $\psi'_m * \psi'_{n-m}$ with $m, n-m \geq 1$ is given by

$$\begin{aligned} \psi'_m * \psi'_{n-m} = &-K_1^R c_1\,|0\rangle * |m\rangle * B_1^L c_1\,|0\rangle * |n-m+1\rangle * K_1^L c_1\,|0\rangle \\ &+ K_1^R c_1\,|0\rangle * |m+1\rangle * B_1^L c_1\,|0\rangle * |n-m\rangle * K_1^L c_1\,|0\rangle\;. \end{aligned} \tag{3.47}$$

Note the simple dependence on m. Therefore,

$$\begin{aligned} \sum_{m=1}^{n-1} \psi'_m * \psi'_{n-m} = &-K_1^R c_1\,|0\rangle * B_1^L c_1\,|0\rangle * |n\rangle * K_1^L c_1\,|0\rangle \\ &+ K_1^R c_1\,|0\rangle * |n\rangle * B_1^L c_1\,|0\rangle * K_1^L c_1\,|0\rangle \end{aligned} \tag{3.48}$$

for $n \geq 2$. The string product $\psi'_0 * \psi'_n$ with $n \geq 1$ is given by

$$\begin{aligned}
\psi'_0 * \psi'_n =& - K_1^R c_1 |0\rangle * B_1^R c_1 |0\rangle * |n\rangle * K_1^L c_1 |0\rangle - B_1^R c_0 c_1 |0\rangle * B_1^R c_1 |0\rangle * |n\rangle * K_1^L c_1 |0\rangle \\
=& - K_1^R c_1 |0\rangle * |n+1\rangle * K_1^L c_1 |0\rangle + K_1^R c_1 |0\rangle * B_1^L c_1 |0\rangle * |n\rangle * K_1^L c_1 |0\rangle \\
& - B_1^R c_0 c_1 |0\rangle * |n+1\rangle * K_1^L c_1 |0\rangle + B_1^R c_0 c_1 |0\rangle * B_1^L c_1 |0\rangle * |n\rangle * K_1^L c_1 |0\rangle \\
=& \, c_1 |0\rangle * |n+1\rangle * (K_1^L)^2 c_1 |0\rangle + K_1^R c_1 |0\rangle * B_1^L c_1 |0\rangle * |n\rangle * K_1^L c_1 |0\rangle \\
& + c_0 c_1 |0\rangle * |n+1\rangle * B_1^L K_1^L c_1 |0\rangle - c_0 c_1 |0\rangle * (B_1^L)^2 c_1 |0\rangle * |n\rangle * K_1^L c_1 |0\rangle \\
=& \, K_1^R c_1 |0\rangle * B_1^L c_1 |0\rangle * |n\rangle * K_1^L c_1 |0\rangle \\
& + c_0 c_1 |0\rangle * |n+1\rangle * B_1^L K_1^L c_1 |0\rangle + c_1 |0\rangle * |n+1\rangle * (K_1^L)^2 c_1 |0\rangle \, . \qquad (3.49)
\end{aligned}$$

The string field $\psi'_n * \psi'_0$ with $n \geq 1$ is given by

$$\begin{aligned}
\psi'_n * \psi'_0 =& - K_1^R c_1 |0\rangle * |n\rangle * B_1^L c_1 |0\rangle * K_1^L c_1 |0\rangle - K_1^R c_1 |0\rangle * |n\rangle * B_1^L c_1 |0\rangle * B_1^L c_0 c_1 |0\rangle \\
=& - K_1^R c_1 |0\rangle * |n\rangle * B_1^L c_1 |0\rangle * K_1^L c_1 |0\rangle \\
& - K_1^R c_1 |0\rangle * |n+1\rangle * B_1^L c_0 c_1 |0\rangle + K_1^R c_1 |0\rangle * |n\rangle * B_1^R c_1 |0\rangle * B_1^L c_0 c_1 |0\rangle \\
=& - K_1^R c_1 |0\rangle * |n\rangle * B_1^L c_1 |0\rangle * K_1^L c_1 |0\rangle \\
& - K_1^R c_1 |0\rangle * |n+1\rangle * B_1^L c_0 c_1 |0\rangle + K_1^R c_1 |0\rangle * |n\rangle * c_1 |0\rangle * (B_1^L)^2 c_0 c_1 |0\rangle \\
=& - K_1^R c_1 |0\rangle * |n\rangle * B_1^L c_1 |0\rangle * K_1^L c_1 |0\rangle + c_1 |0\rangle * |n+1\rangle * B_1^L K_1^L c_0 c_1 |0\rangle \, . \qquad (3.50)
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
\sum_{m=0}^{n} \psi'_m * \psi'_{n-m} =& \, c_0 c_1 |0\rangle * |n+1\rangle * B_1^L K_1^L c_1 |0\rangle + c_1 |0\rangle * |n+1\rangle * (K_1^L)^2 c_1 |0\rangle \\
& + c_1 |0\rangle * |n+1\rangle * B_1^L K_1^L c_0 c_1 |0\rangle \qquad (3.51)
\end{aligned}$$

for $n \geq 1$. We have thus shown (3.2) for $n \geq 1$ by combining this with (3.44). Finally, the string product $\psi'_0 * \psi'_0$ is given by

$$\begin{aligned}
\psi'_0 * \psi'_0 =& - K_1^R c_1 |0\rangle * K_1^L c_1 |0\rangle - K_1^R c_1 |0\rangle * B_1^L c_0 c_1 |0\rangle \\
& - B_1^R c_0 c_1 |0\rangle * K_1^L c_1 |0\rangle - B_1^R c_0 c_1 |0\rangle * B_1^L c_0 c_1 |0\rangle \\
=& \, c_1 |0\rangle * (K_1^L)^2 c_1 |0\rangle + c_1 |0\rangle * B_1^L K_1^L c_0 c_1 |0\rangle \\
& + c_0 c_1 |0\rangle * B_1^L K_1^L c_1 |0\rangle + c_0 c_1 |0\rangle * (B_1^L)^2 c_0 c_1 |0\rangle \\
=& \, c_0 c_1 |0\rangle * B_1^L K_1^L c_1 |0\rangle + c_1 |0\rangle * (K_1^L)^2 c_1 |0\rangle + c_1 |0\rangle * B_1^L K_1^L c_0 c_1 |0\rangle \, , \qquad (3.52)
\end{aligned}$$

and thus $Q_B \, \psi'_1 = -\psi'_0 * \psi'_0$. This completes the proof of (3.2) for the whole range $n \geq 0$.

We have demonstrated that (3.1) and (3.2) can be shown from the set of the equations from (3.24) to (3.33). The first two equations are general properties of the BRST operator and requisites for string field theory to make sense. The operator c_0 only appears in the fourth equation and can be eliminated if we replace $-c_0 c_1 |0\rangle$ by $Q_B c_1 |0\rangle$. Similarly, K_1^L and K_1^R can also be eliminated if we replace them by $\{Q_B, B_1^L\}$ and $\{Q_B, B_1^R\}$, respectively. Now the solution and the proof can be written in terms of $|0\rangle$, Q_B, B_1^L, B_1^R, and c_1, and the equations we need are (3.26), (3.28), (3.29), (3.30), and (3.31). We can construct a different solution if we find a different set of these ingredients which also satisfy the reduced set of the equations shown above. A rather trivial class of replacements is $|0\rangle \to e^A |0\rangle$, $B_1^L \to e^A B_1^L e^{-A}$, $B_1^R \to e^A B_1^R e^{-A}$, and $c_1 \to e^A c_1 e^{-A}$, where A is a linear combination of $K_n = L_n - (-1)^n L_{-n}$. Since A is a derivation of the star product and commutes with Q_B, the solution Ψ_λ is replaced by $e^A \Psi_\lambda$, which obviously satisfies the equation of motion. It also seems possible to replace only B_1^L and B_1^R by different half integrals of the b ghost. There will be many other ways to modify the solutions, while the reduced set of equations are satisfied. However, it is not clear if we can obtain inequivalent solutions in this way.

3.3 Solution in the half-string picture

Schnabl's solution can be naturally described in the half-string picture, where a string field is considered as an operator acting on the Hilbert space of the half string [1, 32–45]. Witten's star product then corresponds to the multiplication of the operators. There is a subtle issue on how to deal with the open string midpoint in this formalism, but we do not attempt to make it rigorous here. We would rather use the formalism to help gain more intuition in manipulating the solution. It is always possible to translate the insight we get in this half-string picture into other languages in previous subsections.

Let us denote the operator associated with the vacuum state $|0\rangle$ by $e^{\frac{\pi}{2}K}$:

$$|0\rangle \sim e^{\frac{\pi}{2}K} . \tag{3.53}$$

This can be thought of as a definition of K. The wedge state $|n\rangle$ then corresponds to

$$|n\rangle \sim e^{\frac{\pi(n-1)}{2}K} . \tag{3.54}$$

The notation is motivated by the formula (3.14). In fact, the operator K here corresponds to an insertion of K defined in (2.26). The state $c_1 |0\rangle$ corresponds to $c(0)$ in the state-operator mapping. The upper half of the unit disk with the insertion $c(0)$ at the origin in the coordinate ξ is mapped by the conformal transformation $z = f_\infty(\xi) = \arctan \xi$ to a semi-infinite strip with a width of $\pi/2$ which has an insertion of $c(0)$ at the origin. Each of the right and left halves of the strip corresponds to $e^{\frac{\pi}{4}K}$. The state $c_1 |0\rangle$ can therefore be expressed as

$$c_1 |0\rangle \sim e^{\frac{\pi}{4}K}\, c\, e^{\frac{\pi}{4}K} , \tag{3.55}$$

where c corresponds to a c-ghost insertion at the end of the half string.

If we denote the operator corresponding to the string field $|\phi\rangle$ by Φ, the actions of B_1^L and B_1^R can be written using a Grassmann-odd operator B as

$$B_1^L |\phi\rangle \sim B\,\Phi , \qquad B_1^R |\phi\rangle \sim -(-1)^\Phi \Phi\, B , \tag{3.56}$$

where $(-1)^{\Phi} = 1$ when Φ is Grassmann even and $(-1)^{\Phi} = -1$ when Φ is Grassmann odd. The operator B here corresponds to an insertion of B defined in (2.14). The relation (3.28) is nothing but the associativity of the operator multiplication in the half-string picture. Similarly,

$$K_1^L \, |\phi\rangle \,\sim\, K\,\Phi\,, \qquad K_1^R \, |\phi\rangle \,\sim\, -\,\Phi\,K\,. \tag{3.57}$$

Note that the operator K here coincides with the one appeared in the operator corresponding to $|0\rangle$.

We have introduced the operators K, B, and c without providing their detailed definitions. All we need is the following set of commutation relations:

$$[\,K, B\,] = 0\,, \qquad \{\,B, c\,\} = 1\,, \qquad c^2 = 0\,, \qquad B^2 = 0\,. \tag{3.58}$$

Note that $[\,K, c\,] \neq 0$.

The action of the BRST operator Q_B is a derivation with respect to the operator multiplication. Let us denote it by d:

$$Q_B \, |\phi\rangle \,\sim\, d\,\Phi\,. \tag{3.59}$$

The derivation property of the Grassmann-odd operation d can be expressed as

$$d\,(\,\Phi_1\,\Phi_2\,) = (\,d\,\Phi_1\,)\,\Phi_2 + (-1)^{\Phi_1}\Phi_1\,(\,d\,\Phi_2\,) \tag{3.60}$$

for any pair of operators Φ_1 and Φ_2, where $(-1)^{\Phi_1} = 1$ when Φ_1 is Grassmann even and $(-1)^{\Phi_1} = -1$ when Φ_1 is Grassmann odd. The actions of d on B, K, and c are given by

$$d\,B = K\,, \qquad d\,K = 0\,, \qquad d\,c = c\,K\,c\,. \tag{3.61}$$

The last equation follows from (2.31) and (3.5). It is easy to verify that $d^2 = 0$ for any operator made of K, B, and c.

The string field ψ'_n corresponds to

$$\psi'_n \,\sim\, e^{\frac{\pi}{4}K}\,c\,B\,K\,e^{\frac{\pi\,n}{2}K}\,c\,e^{\frac{\pi}{4}K}\,. \tag{3.62}$$

This can be easily seen from figure 3. The solution Ψ_λ in (3.3) can be formally written as

$$\Psi_\lambda \,\sim\, \lambda\,e^{\frac{\pi}{4}K}\,c\,\frac{B\,K}{1-\lambda\,e^{\frac{\pi}{2}K}}\,c\,e^{\frac{\pi}{4}K} = f(K)\,c\,\frac{B\,K}{1-f(K)^2}\,c\,f(K)\,, \tag{3.63}$$

where

$$f(K) = \sqrt{\lambda}\,e^{\frac{\pi}{4}K}\,. \tag{3.64}$$

Let us prove that $Q_B\Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0$ in the half-string picture. The string field $Q_B\Psi_\lambda$ corresponds to

$$\begin{aligned} Q_B\Psi_\lambda \sim d\left[\,f(K)\,c\,\frac{B\,K}{1-f(K)^2}\,c\,f(K)\right] \\ = f(K)\,c\,K\,c\,\frac{B\,K}{1-f(K)^2}\,c\,f(K) - f(K)\,c\,\frac{K^2}{1-f(K)^2}\,c\,f(K) \\ +f(K)\,c\,\frac{B\,K}{1-f(K)^2}\,c\,K\,c\,f(K)\,. \end{aligned} \tag{3.65}$$

The string field $\Psi_\lambda * \Psi_\lambda$ corresponds to

$$\begin{aligned}\Psi_\lambda * \Psi_\lambda &\sim f(K)\, c\, \frac{B\, K}{1-f(K)^2}\, c\, f(K)^2\, c\, \frac{B\, K}{1-f(K)^2}\, c\, f(K) \\ &= f(K)\, c\, K\, \frac{1}{1-f(K)^2}\, B\, c\, f(K)^2\, c\, B\, \frac{1}{1-f(K)^2}\, K\, c\, f(K)\,. \end{aligned} \tag{3.66}$$

Since

$$\begin{aligned} B\, c\, f(K)^2\, c\, B &= B\, c\, f(K)^2\, (1 - B\, c) = B\, c\, f(K)^2 - B\, c\, B\, f(K)^2\, c \\ &= B\, c\, f(K)^2 - B\, f(K)^2\, c = B\, c\, f(K)^2 - f(K)^2\, B\, c = [\, B\, c\,,\, f(K)^2\,]\,, \end{aligned} \tag{3.67}$$

$\Psi_\lambda * \Psi_\lambda$ is given by

$$\begin{aligned}\Psi_\lambda * \Psi_\lambda &\sim f(K)\, c\, K\, \frac{1}{1-f(K)^2}\, [\, B\, c\,,\, f(K)^2\,]\, \frac{1}{1-f(K)^2}\, K\, c\, f(K) \\ &= f(K)\, c\, K\, [\, B\, c\,,\, \frac{1}{1-f(K)^2}\,]\, K\, c\, f(K) \\ &= f(K)\, c\, K\, B\, c\, \frac{1}{1-f(K)^2}\, K\, c\, f(K) - f(K)\, c\, K\, \frac{1}{1-f(K)^2}\, B\, c\, K\, c\, f(K) \\ &= -\, f(K)\, c\, K\, c\, \frac{B\, K}{1-f(K)^2}\, c\, f(K) + f(K)\, c\, \frac{K^2}{1-f(K)^2}\, c\, f(K) \\ &\quad -\, f(K)\, c\, \frac{B\, K}{1-f(K)^2}\, c\, K\, c\, f(K)\,. \end{aligned} \tag{3.68}$$

We have thus shown that $Q_B \Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0$. If we expand $1/(1-f(K)^2)$ in powers of $e^{\frac{\pi}{2}K}$, the proof in subsection 3.1 is reproduced. If we instead expand $K/(1-f(K)^2)$ in powers of K when $\lambda = 1$, it will formally give a proof that the solution written in terms of the Bernoulli numbers (1.13) satisfies the equation of motion. However, the expansion in the form of (1.13) will not converge so it is not clear whether the proof for this form is useful.

The proof can be further simplified and made more symmetric if we note

$$\begin{aligned} B\, c\, f(K)^2\, c\, B &= -\, B\, c\, (\, 1 - f(K)^2\,)\, c\, B = -\, (1 - c\, B)\, (\, 1 - f(K)^2\,)\, (1 - B\, c) \\ &= -\, (\, 1 - f(K)^2\,) + (\, 1 - f(K)^2\,)\, B\, c + c\, B\, (\, 1 - f(K)^2\,) \\ &= -\, (\, 1 - f(K)^2\,) + (\, 1 - f(K)^2\,)\, (1 - c\, B) + (1 - B\, c)\, (\, 1 - f(K)^2\,) \\ &= (\, 1 - f(K)^2\,) - (\, 1 - f(K)^2\,)\, c\, B - B\, c\, (\, 1 - f(K)^2\,)\,. \end{aligned} \tag{3.69}$$

We then immediately obtain

$$\begin{aligned}\Psi_\lambda * \Psi_\lambda &\sim f(K)\, c\, K\, \frac{1}{1-f(K)^2}\, B\, c\, f(K)^2\, c\, B\, \frac{1}{1-f(K)^2}\, K\, c\, f(K) \\ &= f(K)\, c\, \frac{K^2}{1-f(K)^2}\, c\, f(K) - f(K)\, c\, K\, c\, \frac{B\, K}{1-f(K)^2}\, c\, f(K) \\ &\quad -\, f(K)\, c\, \frac{B\, K}{1-f(K)^2}\, c\, K\, c\, f(K)\,. \end{aligned} \tag{3.70}$$

Note that we have never used the explicit form of $f(K)$. Therefore, we can formally construct a solution for any choice of the function $f(K)$. It is an important open problem to understand when solutions are well defined and when they become inequivalent.

3.4 Solution as a pure-gauge configuration

The first piece of Schnabl's solution Ψ in (1.5) can be formally written as a pure-gauge configuration $e^{-\Lambda}\,(Q_B\, e^{\Lambda}\,)$ with some gauge parameter Λ. Here and in what follows in this subsection products of string fields are defined using the star product even when the star symbol is omitted.

As we have seen in (3.42), the string field ψ'_0 is BRST exact:

$$\psi'_0 = Q_B\,\Phi\,, \tag{3.71}$$

where

$$\Phi = B_1^L\, c_1\,|0\rangle\;. \tag{3.72}$$

A crucial observation is that the string field ψ'_n for integer n with $n \geq 1$ can also be written in terms of Q_B and Φ:

$$\psi'_n = (Q_B\,\Phi)\,\Phi^n\,. \tag{3.73}$$

It is easy to see in the CFT formulation that $\psi'_n = \psi'_0\,\Phi^n$ by repeatedly using the relation $B\,c(z)\,B = B$. It can also be shown in the following way. Using (3.31), (3.28), and (3.29), we find that

$$\Phi^2 = |0\rangle * \Phi\,. \tag{3.74}$$

Therefore,

$$\Phi^n = \underbrace{|0\rangle * |0\rangle * \ldots * |0\rangle}_{n-1} * \Phi = |n\rangle * \Phi\,. \tag{3.75}$$

Since

$$Q_B\,\Phi = \psi'_0 = \; - K_1^R c_1\,|0\rangle - B_1^R c_0 c_1\,|0\rangle\;, \tag{3.76}$$

we obtain

$$(Q_B\,\Phi)\,\Phi^n = \; - K_1^R c_1\,|0\rangle * |n\rangle * B_1^L\,c_1\,|0\rangle = c_1\,|0\rangle * |n\rangle * B_1^L\,K_1^L\,c_1\,|0\rangle = \psi'_n\,. \tag{3.77}$$

The solution Ψ_λ can be written as

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1}\,\psi'_n = \sum_{n=0}^{\infty} \lambda^{n+1}\,(Q_B\,\Phi)\,\Phi^n = \lambda\,(Q_B\,\Phi)\,\frac{1}{1-\lambda\,\Phi}\,. \tag{3.78}$$

Since $e^{-\Lambda}\,(Q_B\,e^{\Lambda}\,) = -(Q_B\,e^{-\Lambda}\,)\,e^{\Lambda}$, the gauge parameter Λ can be written in terms of Φ as follows:

$$e^{\Lambda} = \frac{1}{1-\lambda\,\Phi}\,, \tag{3.79}$$

or

$$\Lambda = -\ln\,(1-\lambda\,\Phi) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n}\Phi^n\,. \tag{3.80}$$

It is now straightforward to see that Ψ_λ satisfies the equation of motion. Since

$$Q_B \frac{1}{1-\lambda\,\Phi} = \frac{1}{1-\lambda\,\Phi}\,\lambda\,(Q_B\,\Phi)\,\frac{1}{1-\lambda\,\Phi}\,, \tag{3.81}$$

$Q_B\,\Psi_\lambda$ is given by

$$Q_B\,\Psi_\lambda = \,-\,\lambda\,(Q_B\,\Phi)\,Q_B\frac{1}{1-\lambda\,\Phi} = \,-\,\lambda\,(Q_B\,\Phi)\,\frac{1}{1-\lambda\,\Phi}\,\lambda\,(Q_B\,\Phi)\,\frac{1}{1-\lambda\,\Phi} = -\Psi_\lambda^2\,. \quad (3.82)$$

Therefore,

$$Q_B\,\Psi_\lambda + \Psi_\lambda^2 = 0\,. \quad (3.83)$$

It is also straightforward to calculate $Q_B\,\psi'_n$ using the expression (3.73):

$$\begin{aligned} Q_B\,\psi'_n &= Q_B\,[\,(Q_B\,\Phi)\,\Phi^n\,] = \,-\,(Q_B\,\Phi)\,(Q_B\,\Phi^n) = \,-\,\sum_{m=0}^{n-1}(Q_B\,\Phi)\,\Phi^m\,(Q_B\,\Phi)\,\Phi^{n-m-1} \\ &= \,-\,\sum_{m=0}^{n-1}\psi'_m\,\psi'_{n-m-1} \end{aligned} \quad (3.84)$$

for $n \ge 1$. We have thus reproduced (3.2).

The string field ψ'_0 can also be written as

$$\psi'_0 = Q_B\,\widetilde{\Phi}\,, \quad (3.85)$$

where

$$\widetilde{\Phi} = \,-\,B_1^R c_1\,|0\rangle\;. \quad (3.86)$$

The string field ψ'_n for integer n with $n \ge 1$ can also be written in terms of Q_B and $\widetilde{\Phi}$:

$$\psi'_n = (-1)^n\,\widetilde{\Phi}^n\,(Q_B\,\widetilde{\Phi})\,. \quad (3.87)$$

More generally, it can be written using both Φ and $\widetilde{\Phi}$ as

$$\psi'_n = (-1)^m\,\widetilde{\Phi}^m\,(Q_B\,\Phi)\,\Phi^{n-m} = (-1)^m\,\widetilde{\Phi}^m\,(Q_B\,\widetilde{\Phi})\,\Phi^{n-m}\,, \quad (3.88)$$

for any integer m with $0 \le m \le n$.

4. Kinetic term

The kinetic term of the Witten's string field theory action was evaluated for Schnabl's solution in [13]:

$$\mathcal{K}_2 = \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{N}\langle\,\psi'_n, Q_B\psi'_m\,\rangle = \frac{1}{2} - \frac{1}{\pi^2}\,, \quad (4.1)$$

$$\mathcal{K}_1 = \lim_{N\to\infty}\sum_{m=0}^{N}\langle\,\psi_N, Q_B\psi'_m\,\rangle = \frac{1}{2} + \frac{2}{\pi^2}\,, \quad (4.2)$$

$$\mathcal{K}_0 = \lim_{N\to\infty}\langle\,\psi_N, Q_B\psi_N\,\rangle = \frac{1}{2} + \frac{2}{\pi^2}\,. \quad (4.3)$$

The subscript of $\mathcal{K}$ indicates the number of sums. We reproduce these results in a different way in this section. Our way of calculating these quantities makes it clear how the ψ_N

piece cancels the difference between $\mathcal{K}_2$ and the value (1.3) predicted by Sen's conjecture. Our method also has an advantage in generalizing to the calculations for the cubic term in the next section.

The inner product $\langle\, \psi_n, Q_B \psi_m \,\rangle$ was calculated in [13]:

$$\begin{aligned}
&\langle\, \psi_n, Q_B \psi_m \,\rangle \\
&= \frac{1}{\pi^2}\left(1+\cos\frac{\pi\,(m-n)}{m+n+2}\right)\left(-1+\frac{m+n+2}{\pi}\sin\frac{2\,\pi}{m+n+2}\right) \\
&\quad +2\sin^2\frac{\pi}{m+n+2}\Bigg[-\frac{m+n+1}{\pi^2}+\frac{mn}{\pi^2}\cos\frac{\pi\,(m-n)}{m+n+2} \\
&\qquad\qquad +\frac{(m+n+2)(m-n)}{2\,\pi^3}\sin\frac{\pi\,(m-n)}{m+n+2}\Bigg]. \qquad (4.4)
\end{aligned}$$

Using this expression, it was shown in [13] that

$$\sum_{m=0}^{n} \langle\, \psi'_m, Q_B \psi'_{n-m} \,\rangle = 0\,. \qquad (4.5)$$

The double sum in $\mathcal{K}_2$ before taking the limit $N \to \infty$ can be decomposed in the following way:

$$\begin{aligned}
\sum_{n=0}^{N}\sum_{m=0}^{N} \langle\, \psi'_n, Q_B \psi'_m \,\rangle &= \sum_{m=0}^{N}\sum_{n=0}^{N-m} \langle\, \psi'_n, Q_B \psi'_m \,\rangle + \sum_{m=1}^{N}\sum_{n=N-m+1}^{N} \langle\, \psi'_n, Q_B \psi'_m \,\rangle \\
&= \sum_{n=0}^{N}\sum_{m=0}^{n} \langle\, \psi'_m, Q_B \psi'_{n-m} \,\rangle + \sum_{m=1}^{N}\sum_{n=N-m+1}^{N} \langle\, \psi'_n, Q_B \psi'_m \,\rangle\,. \quad (4.6)
\end{aligned}$$

The first double sum in the last line vanishes because of (4.5). Therefore, $\mathcal{K}_2$ reduces to

$$\mathcal{K}_2 = \lim_{N\to\infty} \sum_{m=1}^{N}\sum_{n=N-m+1}^{N} \langle\, \psi'_n, Q_B \psi'_m \,\rangle\,. \qquad (4.7)$$

Now all of $\mathcal{K}_2$, $\mathcal{K}_1$, and $\mathcal{K}_0$ are written in terms of inner products of the form $\langle\, \psi_n, Q_B \psi_m \,\rangle$ with large $n+m$ and their derivatives. When $n+m$ is large, the inner product $\langle\, \psi_n, Q_B \psi_m \,\rangle$ becomes

$$\begin{aligned}
&\lim_{n+m\to\infty} \langle\, \psi_n, Q_B \psi_m \,\rangle \\
&= \frac{1}{\pi^2}\left(1+\cos\frac{\pi(m-n)}{m+n}\right)+\frac{2mn}{(m+n)^2}\cos\frac{\pi(m-n)}{m+n}+\frac{m-n}{\pi(m+n)}\sin\frac{\pi\,(m-n)}{m+n}\,. \quad (4.8)
\end{aligned}$$

Note that only the sum $n+m$ needs to be large for this expression to be valid, and either n or m can be small as long as the sum is large. Let us introduce the function

$$\mathcal{F}_K(x,y) = \frac{1}{\pi^2}\left(1+\cos\frac{\pi\,(x-y)}{x+y}\right)+\frac{2\,xy}{(x+y)^2}\cos\frac{\pi\,(x-y)}{x+y}+\frac{x-y}{\pi(x+y)}\sin\frac{\pi\,(x-y)}{x+y}\,. \qquad (4.9)$$

Note that

$$\mathcal{F}_K(0,y) = \mathcal{F}_K(x,0) = 0\,, \tag{4.10}$$

and

$$\mathcal{F}_K(ax,ay) = \mathcal{F}_K(x,y) \tag{4.11}$$

for any nonvanishing a. The inner product (4.8) in the limit can be written as

$$\lim_{n+m\to\infty} \langle\, \psi_n, Q_B\psi_m \,\rangle = \mathcal{F}_K\,(an,am) \tag{4.12}$$

for any nonvanishing a. In particular, we can choose a to be $1/N$ in taking the limit $N \to \infty$:

$$\lim_{n+m\to\infty} \langle\, \psi_n, Q_B\psi_m \,\rangle = \mathcal{F}_K\left(\frac{n}{N},\frac{m}{N}\right)\,. \tag{4.13}$$

The quantity $\mathcal{K}_0$ is simply given by

$$\mathcal{K}_0 = \lim_{N\to\infty} \langle\, \psi_N, Q_B\psi_N \,\rangle = \mathcal{F}_K(1,1) = \frac{2}{\pi^2}+\frac{1}{2}\,. \tag{4.14}$$

The quantity $\mathcal{K}_1$ can be written as

$$\mathcal{K}_1 = \lim_{N\to\infty}\sum_{n=0}^{N} \langle\, \psi'_n, Q_B\psi_N \,\rangle = \lim_{N\to\infty}\sum_{n=0}^{N} \frac{\partial}{\partial n}\mathcal{F}_K\left(\frac{n}{N},1\right) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\frac{\partial}{\partial x}\mathcal{F}_K\,(x,1)\bigg|_{x=\frac{n}{N}}\,. \tag{4.15}$$

Using the formula

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N} f\left(\frac{n}{N}\right) = \int_0^1 dx\, f(x)\,, \tag{4.16}$$

we obtain

$$\mathcal{K}_1 = \int_0^1 dx\,\frac{\partial}{\partial x}\mathcal{F}_K\,(x,1) = \mathcal{F}_K(1,1) - \mathcal{F}_K(0,1) = \mathcal{F}_K(1,1) = \frac{2}{\pi^2}+\frac{1}{2}\,, \tag{4.17}$$

where we used that $\mathcal{F}_K(0,y) = 0$. It is more or less obvious in this way of the calculation that $\mathcal{K}_1$ and $\mathcal{K}_0$ coincide, while it was not the case in [13], where $\mathcal{K}_1$ was obtained from a nontrivial integral.

The expression of $\mathcal{K}_2$ in (4.7) can also be transformed into an integral:

$$\begin{aligned}
\mathcal{K}_2 &= \lim_{N\to\infty}\sum_{m=1}^{N}\sum_{n=N-m+1}^{N}\frac{\partial}{\partial n}\frac{\partial}{\partial m}\mathcal{F}_K\left(\frac{n}{N},\frac{m}{N}\right)\\
&= \lim_{N\to\infty}\frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=N-m+1}^{N}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathcal{F}_K\,(x,y)\bigg|_{x=\frac{n}{N},\,y=\frac{m}{N}}\\
&= \int_0^1 dy\int_{1-y}^1 dx\,\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathcal{F}_K\,(x,y)\;.
\end{aligned} \tag{4.18}$$

The integration over x is trivial:

$$\int_0^1 dy\int_{1-y}^1 dx\,\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathcal{F}_K\,(x,y) = \int_0^1 dy\,\frac{\partial}{\partial y}\mathcal{F}_K\,(1,y) - \int_0^1 dy\,\frac{\partial}{\partial y}\mathcal{F}_K\,(x,y)\bigg|_{x=1-y}\,. \tag{4.19}$$

The first term is

$$\int_0^1 dy\, \frac{\partial}{\partial y} \mathcal{F}_K\,(1,y) = \mathcal{F}_K\,(1,1) - \mathcal{F}_K\,(1,0) = \mathcal{F}_K(1,1) = \frac{2}{\pi^2} + \frac{1}{2}\,. \tag{4.20}$$

The second term can also be calculated directly:

$$-\int_0^1 dy\, \frac{\partial}{\partial y} \mathcal{F}_K\,(x,y)\bigg|_{x=1-y} = -\,4\pi \int_0^1 dy\;(1-y)^2 y\,\sin 2\,\pi y = -\frac{3}{\pi^2}\,. \tag{4.21}$$

It can also be calculated in the following way. Let us first write

$$-\int_0^1 dy\, \frac{\partial}{\partial y} \mathcal{F}_K\,(x,y)\bigg|_{x=1-y} = -\int_0^1 dt\,\partial_y \mathcal{F}_K\,(x,y)\bigg|_{x=1-t,\,y=t}\,. \tag{4.22}$$

Using the relation

$$[\,x\,\partial_x + y\,\partial_y\,]\,\mathcal{F}_K(x,y) = 0\,, \tag{4.23}$$

which follows from (4.11), and

$$\partial_t \mathcal{F}_K\,(1-t,t) = \;-\,\partial_x \mathcal{F}_K\,(x,y)\bigg|_{x=1-t,\,y=t} + \partial_y \mathcal{F}_K\,(x,y)\bigg|_{x=1-t,\,y=t}\,, \tag{4.24}$$

we find

$$\partial_y \mathcal{F}_K(x,y)\bigg|_{x=1-t,\,y=t} = (1-t)\,\partial_t\,\mathcal{F}_K(1-t,t) = [\,\partial_t\,(1-t)+1\,]\,\mathcal{F}_K(1-t,t)\,. \tag{4.25}$$

Since

$$-\int_0^1 dt\,\partial_t\Big[\,(1-t)\,\mathcal{F}_K\,(1-t,t)\,\Big] = \mathcal{F}_K(1,0) = 0\,, \tag{4.26}$$

we obtain

$$-\int_0^1 dy\, \frac{\partial}{\partial y} \mathcal{F}_K\,(x,y)\bigg|_{x=1-y} = -\int_0^1 dt\,\mathcal{F}_K\,(1-t,t)\;. \tag{4.27}$$

It is straightforward to carry out the integral:

$$\begin{aligned} &-\int_0^1 dt\,\mathcal{F}_K\,(1-t,t) \\ &= -\int_0^1 dt \left[\frac{1}{\pi^2}\Big(1+\cos(\pi-2\pi t)\Big) + 2(1-t)t\cos(\pi-2\pi t) + \frac{1-2t}{\pi}\sin(\pi-2\pi t)\right] \\ &= -\frac{3}{\pi^2}\,. \end{aligned} \tag{4.28}$$

The quantity $\mathcal{K}_2$ is therefore given by

$$\mathcal{K}_2 = \mathcal{F}_K(1,1) - \frac{3}{\pi^2} = \frac{1}{2} - \frac{1}{\pi^2}\,. \tag{4.29}$$

The difference between $\mathcal{K}_2$ and the value (1.3) predicted from Sen's conjecture is written in terms of $\mathcal{F}_K(1,1)$, and it is easy to see the way it is canceled in (1.12) from our method of the calculation.

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5. Cubic term

In order to evaluate the cubic term of the Witten's string field theory action for Schnabl's solution (1.5), we need to calculate the following quantities:

$$\mathcal{V}_3 = \lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{N} \sum_{k=0}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle\,, \tag{5.1}$$

$$\mathcal{V}_2 = \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{k=0}^{N} \langle\, \psi_N, \psi'_m * \psi'_k \,\rangle\,, \tag{5.2}$$

$$\mathcal{V}_1 = \lim_{N\to\infty} \sum_{n=0}^{N} \langle\, \psi'_n, \psi_N * \psi_N \,\rangle\,, \tag{5.3}$$

$$\mathcal{V}_0 = \lim_{N\to\infty} \langle\, \psi_N, \psi_N * \psi_N \,\rangle\,, \tag{5.4}$$

where the subscript of $\mathcal{V}$ again indicates the number of sums. These quantities all consist of inner products of the form $\langle\, \psi_n, \psi_m * \psi_k \,\rangle$ and their derivatives. We calculate the inner product $\langle\, \psi_n, \psi_m * \psi_k \,\rangle$ in the first subsection and then calculate the summations in the second subsection.

5.1 Correlation functions

The inner product $\langle\, \psi_n, \psi_m * \psi_k \,\rangle$ is given by

$$\begin{aligned} &\langle\, \psi_n, \psi_m * \psi_k \,\rangle \\ &= \left(\frac{2}{\pi}\right)^3 \langle\, B\, c(x-a)\, c(x)\, B\, c(x+y-a)\, c(x+y)\, B\, c(x+y+z-a)\, c(x+y+z) \,\rangle_{C_{x+y+z}} \end{aligned} \tag{5.5}$$

with

$$x = \frac{\pi n}{2} + \frac{\pi}{2}\,, \qquad y = \frac{\pi m}{2} + \frac{\pi}{2}\,, \qquad z = \frac{\pi k}{2} + \frac{\pi}{2}\,, \qquad a = \frac{\pi}{2}\,. \tag{5.6}$$

See figure 4. The calculation of this correlation function can be reduced to that of the three-point function of c ghosts on the same semi-infinite cylinder, as we will demonstrate below. The three-point function of c ghosts on the semi-infinite cylinder C_π can be obtained from (2.8) by the conformal transformation $z = f_\infty(w) = \arctan w$ and is given by

$$\langle\, c(z_1)\, c(z_2)\, c(z_3) \,\rangle_{C_\pi} = \sin(z_1 - z_2)\sin(z_1 - z_3)\sin(z_2 - z_3)\,. \tag{5.7}$$

The three-point function of c ghosts on a general semi-infinite cylinder C_n is

$$\langle\, c(z_1)\, c(z_2)\, c(z_3) \,\rangle_{C_n} = \left(\frac{n}{\pi}\right)^3 \sin\frac{\pi(z_1 - z_2)}{n} \sin\frac{\pi(z_1 - z_3)}{n} \sin\frac{\pi(z_2 - z_3)}{n}\,. \tag{5.8}$$

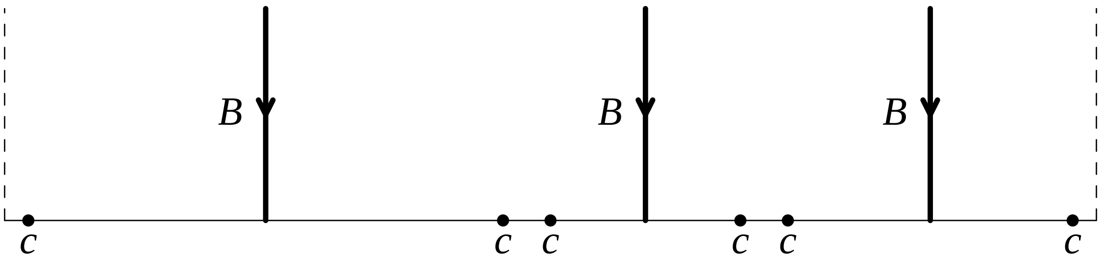

Figure 4: A representation of the inner product $\langle\, \psi_n, \psi_m * \psi_k \,\rangle$. The two dashed lines are identified. Each of the three states is represented by figure 2 with the shaded region deleted. The semi-infinite cylinder in this figure is constructed by gluing together the resulting three surfaces. The c ghost near the left dashed line has been brought to the right in (5.5) using the periodicity.

Let us next calculate the correlation function $\langle\, B\, c(z_1)\, c(z_2)\, c(z_3)\, c(z_4) \,\rangle_{C_\pi}$. One way of calculating this is to carry out the integral of the b ghost in B explicitly, for example, on the upper-half plane. It can also be obtained in the following indirect way. Since

$$\begin{aligned} &\langle\, B\, c(z_1)\, c(z_2)\, c(z_3)\, c(z_4) \,\rangle_{C_\pi} \\ &= \frac{1}{2}\, \langle\, B\, c(z_1)\, c(z_2)\, c(z_3)\, c(z_4) \,\rangle_{C_\pi} + \frac{1}{2}\, \langle\, c(z_1)\, c(z_2)\, c(z_3)\, c(z_4)\, B \,\rangle_{C_\pi}\,, \end{aligned} \tag{5.9}$$

the integral of the b ghost can be written after the conformal transformation $\xi = \tan z$ to the upper-half plane with the coordinate ξ as

$$\frac{1}{2}\left(-B_1^R + B_1^L\right) = -\frac{1}{2} \oint \frac{d\xi}{2\pi i}\, (\xi^2+1)\, \varepsilon(\,\mathrm{Re}\,\xi\,)\, b(\xi)\,, \tag{5.10}$$

where $\mathrm{Re}\,\xi$ is the real part of ξ, and the step function $\varepsilon(x)$ is defined to be $\varepsilon(x) = 1$ for $x > 0$ and $\varepsilon(x) = -1$ for $x < 0$. The contour of the integral should encircle all of the four c ghosts counterclockwise. Furthermore,

$$\frac{1}{2}\left(-B_1^R + B_1^L\right) = -\frac{1}{\pi} \oint \frac{d\xi}{2\pi i}\, (\xi^2+1)\, (\arctan\xi + \mathrm{arccot}\,\xi)\, b(\xi) = -\frac{1}{\pi}\left(\mathcal{B}_0 + \mathcal{B}_0^\dagger\right)\,, \tag{5.11}$$

where

$$\mathcal{B}_0 = \oint \frac{d\xi}{2\pi i}\, (\xi^2+1)\, \arctan\xi\, b(\xi)\,, \tag{5.12}$$

and $\mathcal{B}_0^\dagger$ is its BPZ conjugate. There is no contribution from the $\mathcal{B}_0^\dagger$ part because the integrand is regular at infinity and there are no operator insertions outside the contour. The integrand of $\mathcal{B}_0$ is regular at the origin, but there are four c-ghost insertions inside the contour. Since $c(z)$ is mapped to $\cos^2 z\, c(\tan z)$ in the ξ coordinate, the contribution from each c-ghost insertion is

$$\cos^2 z \left(-\frac{1}{\pi}\right) \oint \frac{d\xi}{2\pi i}\, (\xi^2+1)\, \arctan\xi\, b(\xi)\, c(\tan z) = -\frac{z}{\pi}\,. \tag{5.13}$$

After mapping the upper-half plane back to C_π and taking into account the signs from anticommuting ghosts, the correlation function is given by

$$\begin{aligned}\langle\, B\, c(z_1)\, c(z_2)\, c(z_3)\, c(z_4)\,\rangle_{C_\pi} \;=\; & -\frac{z_1}{\pi}\,\langle\, c(z_2)\, c(z_3)\, c(z_4)\,\rangle_{C_\pi} + \frac{z_2}{\pi}\,\langle\, c(z_1)\, c(z_3)\, c(z_4)\,\rangle_{C_\pi} \\ & -\frac{z_3}{\pi}\langle c(z_1)c(z_2)c(z_4)\rangle_{C_\pi} + \frac{z_4}{\pi}\langle\, c(z_1)c(z_2)\, c(z_3)\,\rangle_{C_\pi}\,. \qquad (5.14)\end{aligned}$$

The correlation function $\langle Bc(z_1)c(z_2)c(z_3)c(z_4)\rangle_{C_n}$ on a general semi-infinite cylinder C_n is

$$\begin{aligned}\langle Bc(z_1)c(z_2)c(z_3)c(z_4)\rangle_{C_n} \;=\; & -\frac{z_1}{n}\langle c(z_2)c(z_3)c(z_4)\rangle_{C_n} + \frac{z_2}{n}\langle c(z_1)c(z_3)c(z_4)\rangle_{C_n} \\ & -\frac{z_3}{n}\langle c(z_1)c(z_2)c(z_4)\rangle_{C_n} + \frac{z_4}{n}\langle c(z_1)c(z_2)c(z_3)\rangle_{C_n}\,. \qquad (5.15)\end{aligned}$$

Using the relation (3.9), the correlation function with three B insertions reduces to

$$\begin{aligned} & \langle\, B\, c(z_1)\, c(z_2)\, B\, c(z_3)\, c(z_4)\, B\, c(z_5)\, c(z_6)\,\rangle_{C_\pi} \\ = & \langle\, B\, c(z_1)\, c(z_2)\, B\, c(z_3)\, c(z_5)\, c(z_6)\,\rangle_{C_\pi} - \langle\, B\, c(z_1)\, c(z_2)\, B\, c(z_4)\, c(z_5)\, c(z_6)\,\rangle_{C_\pi} \\ = & \langle\, B\, c(z_1)\, c(z_3)\, c(z_5)\, c(z_6)\,\rangle_{C_\pi} - \langle\, B\, c(z_2)\, c(z_3)\, c(z_5)\, c(z_6)\,\rangle_{C_\pi} \\ & - \langle\, B\, c(z_1)\, c(z_4)\, c(z_5)\, c(z_6)\,\rangle_{C_\pi} + \langle\, B\, c(z_2)\, c(z_4)\, c(z_5)\, c(z_6)\,\rangle_{C_\pi}\,. \qquad (5.16)\end{aligned}$$

The expression significantly simplifies in the following case which is of our interest:

$$\begin{aligned} & \langle\, B\, c(z_1)\, c(z_1+a)\, B\, c(z_2)\, c(z_2+a)\, B\, c(z_3)\, c(z_3+a)\,\rangle_{C_\pi} \\ = & \frac{4\,a}{\pi}\,\sin^2 a\,\sin(z_1-z_2)\,\sin(z_1-z_3)\,\sin(z_2-z_3)\,, \qquad (5.17)\end{aligned}$$

where we have used the formulas

$$\sin(x+a)\,\sin(y+a) - \sin x\,\sin y = \sin a\,\sin(x+y+a)\,, \qquad (5.18)$$

$$\sin(x+a)\,\sin y - \sin x\,\sin(y+a) = -\sin a\,\sin(x-y)\,, \qquad (5.19)$$

$$\sin(2x-2y)+\sin(2y-2z)+\sin(2z-2x) = 4\sin(x-y)\sin(x-z)\sin(y-z)\,. \qquad (5.20)$$

It is interesting to note that

$$\langle\, B\, c(z_1)\, c(z_1+a)\, B\, c(z_2)\, c(z_2+a)\, B\, c(z_3)\, c(z_3+a)\,\rangle_{C_\pi} = \frac{4\,a}{\pi}\,\sin^2 a\,\langle\, c(z_1)\, c(z_2)\, c(z_3)\,\rangle_{C_\pi}\,. \qquad (5.21)$$

The correlation function on C_n is given by

$$\begin{aligned} & \langle\, B\, c(z_1)\, c(z_1+a)\, B\, c(z_2)\, c(z_2+a)\, B\, c(z_3)\, c(z_3+a)\,\rangle_{C_n} \\ = & \frac{4\,a\,n^2}{\pi^3}\,\sin^2\frac{\pi a}{n}\,\sin\frac{\pi(z_1-z_2)}{n}\,\sin\frac{\pi(z_1-z_3)}{n}\,\sin\frac{\pi(z_2-z_3)}{n}\,, \qquad (5.22)\end{aligned}$$

and the inner product $\langle\, \psi_n, \psi_m * \psi_k\,\rangle$ is

$$\begin{aligned} & \langle\, \psi_n, \psi_m * \psi_k\,\rangle \qquad (5.23) \\ = & -\left(\frac{2}{\pi}\right)^3\,\frac{4\,a\,(x+y+z)^2}{\pi^3}\,\sin^2\frac{\pi a}{x+y+z}\,\sin\frac{\pi x}{x+y+z}\,\sin\frac{\pi y}{x+y+z}\,\sin\frac{\pi z}{x+y+z}\end{aligned}$$

with

$$x = \frac{\pi n}{2} + \frac{\pi}{2}\,, \qquad y = \frac{\pi m}{2} + \frac{\pi}{2}\,, \qquad z = \frac{\pi k}{2} + \frac{\pi}{2}\,, \qquad a = \frac{\pi}{2}\,. \qquad (5.24)$$

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5.2 Summations

In the proofs of (3.1) and (3.2) in section 3, it was assumed that these equations are contracted with a state in the Fock space. However, it is straightforward to see in the CFT formulation, for example, that the proofs can be extended to the case where the equations are contracted with ψ'_k for any k in the range $k \geq 0$:

$$\langle\, \psi'_k, Q_B \psi'_0 \,\rangle = 0 \,, \qquad \langle\, \psi'_k, Q_B \psi'_{n+1} \,\rangle = - \sum_{m=0}^{n} \langle\, \psi'_k, \psi'_m * \psi'_{n-m} \,\rangle \,. \tag{5.25}$$

Combining these with (4.5), we obtain

$$\sum_{m=0}^{n} \sum_{k=0}^{n-m} \langle\, \psi'_m, \psi'_k * \psi'_{n-m-k} \,\rangle = 0 \tag{5.26}$$

for any nonnegative integer n. The triple sum in $\mathcal{V}_3$ before taking the limit $N \to \infty$ can be decomposed in the following way:

$$\begin{aligned}
& \sum_{n=0}^{N} \sum_{m=0}^{N} \sum_{k=0}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle \\
&= \sum_{k=0}^{N} \sum_{m=0}^{N-k} \sum_{n=0}^{N-k-m} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle \\
&\quad + \sum_{k=0}^{N} \sum_{m=0}^{N-k} \sum_{n=N-k-m+1}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle + \sum_{k=1}^{N} \sum_{m=N-k+1}^{N} \sum_{n=0}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle \\
&= \sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{k=0}^{n-m} \langle\, \psi'_m, \psi'_k * \psi'_{n-m-k} \,\rangle \\
&\quad + \sum_{k=0}^{N} \sum_{m=0}^{N-k} \sum_{n=N-k-m+1}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle + \sum_{k=1}^{N} \sum_{m=N-k+1}^{N} \sum_{n=0}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle
\end{aligned} \tag{5.27}$$

with the understanding that there is no contribution to the second triple sum on the right-hand side when $k = m = 0$. The first triple sum on the right-hand side of (5.27) vanishes because of (5.26). The quantity $\mathcal{V}_3$ is thus given by

$$\mathcal{V}_3 = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{m=0}^{N-k} \sum_{n=N-k-m+1}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle + \lim_{N\to\infty} \sum_{k=1}^{N} \sum_{m=N-k+1}^{N} \sum_{n=0}^{N} \langle\, \psi'_n, \psi'_m * \psi'_k \,\rangle \,. \tag{5.28}$$

As in the case of the evaluation of the kinetic term in the previous section, all of $\mathcal{V}_3$, $\mathcal{V}_2$, $\mathcal{V}_1$, and $\mathcal{V}_0$ are now written in terms of inner products of the form $\langle\, \psi_n, \psi_m * \psi_k \,\rangle$ with $n+m+k$ large and their derivatives. When $n+m+k$ is large, the inner product $\langle\, \psi_n, \psi_m * \psi_k \,\rangle$ is given by

$$\lim_{n+m+k\to\infty} \langle\, \psi_n, \psi_m * \psi_k \,\rangle = -\frac{4}{\pi} \sin\frac{\pi n}{n+m+k} \sin\frac{\pi m}{n+m+k} \sin\frac{\pi k}{n+m+k} \,. \tag{5.29}$$

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Let us introduce the function

$$\mathcal{F}_V(x,y,z) = -\frac{4}{\pi}\sin\frac{\pi x}{x+y+z}\sin\frac{\pi y}{x+y+z}\sin\frac{\pi z}{x+y+z}\,. \tag{5.30}$$

Note that

$$\mathcal{F}_V(0,y,z) = \mathcal{F}_V(x,0,z) = \mathcal{F}_V(x,y,0) = 0 \tag{5.31}$$

and

$$\mathcal{F}_V(ax,ay,az) = \mathcal{F}_V(x,y,z) \tag{5.32}$$

for any nonvanishing a. Then the inner product $\langle\,\psi_n, \psi_m * \psi_k\,\rangle$ in the limit $n+m+k \to \infty$ can be written as

$$\lim_{n+m+k\to\infty}\langle\,\psi_n, \psi_m * \psi_k\,\rangle = \mathcal{F}_V\,(an, am, ak) \tag{5.33}$$

for any nonvanishing a. We can in particular choose a to be $1/N$ in taking the limit $N \to \infty$:

$$\lim_{n+m+k\to\infty}\langle\,\psi_n, \psi_m * \psi_k\,\rangle = \mathcal{F}_V\left(\frac{n}{N},\frac{m}{N},\frac{k}{N}\right)\,. \tag{5.34}$$

The quantity $\mathcal{V}_0$ is readily given by

$$\mathcal{V}_0 = \lim_{N\to\infty}\langle\,\psi_N, \psi_N * \psi_N\,\rangle = \mathcal{F}_V(1,1,1) = -\frac{3\sqrt{3}}{2\,\pi}\,. \tag{5.35}$$

The sums in $\mathcal{V}_1$ and $\mathcal{V}_2$ can be transformed into integrals. The quantity $\mathcal{V}_1$ is given by

$$\begin{aligned}\mathcal{V}_1 &= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{\partial}{\partial n}\langle\,\psi_n, \psi_N * \psi_N\,\rangle = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{\partial}{\partial n}\mathcal{F}_V\left(\frac{n}{N},1,1\right)\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\frac{\partial}{\partial x}\mathcal{F}_V\,(x,1,1)\bigg|_{x=\frac{n}{N}} = \int_0^1 dx\,\frac{\partial}{\partial x}\mathcal{F}_V\,(x,1,1)\\ &= \mathcal{F}_V(1,1,1) = -\frac{3\sqrt{3}}{2\,\pi}\,,\end{aligned} \tag{5.36}$$

and $\mathcal{V}_2$ is

$$\begin{aligned}\mathcal{V}_2 &= \lim_{N\to\infty}\sum_{m=0}^{N}\sum_{k=0}^{N}\frac{\partial}{\partial m}\frac{\partial}{\partial k}\langle\,\psi_N, \psi_m * \psi_k\,\rangle = \lim_{N\to\infty}\sum_{m=0}^{N}\sum_{k=0}^{N}\frac{\partial}{\partial m}\frac{\partial}{\partial k}\mathcal{F}_V\left(1,\frac{m}{N},\frac{k}{N}\right)\\ &= \lim_{N\to\infty}\frac{1}{N^2}\sum_{m=0}^{N}\sum_{k=0}^{N}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(1,y,z)\bigg|_{y=\frac{m}{N},\,z=\frac{k}{N}}\\ &= \int_0^1 dy\int_0^1 dz\,\frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(1,y,z) = \int_0^1 dy\,\frac{\partial}{\partial y}\mathcal{F}_V\,(1,y,1) = \mathcal{F}_V(1,1,1) = -\frac{3\sqrt{3}}{2\,\pi}\,.\end{aligned} \tag{5.37}$$

Finally, $\mathcal{V}_3$ in the form of (5.28) can be expressed as a sum of two integrals:

$$\begin{aligned}\mathcal{V}_3 &= \int_0^1 dz\int_0^{1-z} dy\int_{1-z-y}^1 dx\,\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(x,y,z)\\ &\quad+\int_0^1 dz\int_{1-z}^1 dy\int_0^1 dx\,\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(x,y,z)\,.\end{aligned} \tag{5.38}$$

The integration over x is trivial:

$$\begin{aligned}\mathcal{V}_3 =& \int_0^1 dz \int_0^{1-z} dy \, \frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(1,y,z) - \int_0^1 dz \int_0^{1-z} dy \, \frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(x,y,z)\Big|_{x=1-z-y} \\ &+ \int_0^1 dz \int_{1-z}^1 dy \, \frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(1,y,z) \\ =& \int_0^1 dz \int_0^1 dy \frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(1,y,z) - \int_0^1 dz \int_0^{1-z} dy \frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(x,y,z)\Big|_{x=1-z-y} . \end{aligned} \tag{5.39}$$

The first term is

$$\int_0^1 dz \int_0^1 dy \, \frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(1,y,z) = \mathcal{F}_V(1,1,1) = -\frac{3\sqrt{3}}{2\,\pi} . \tag{5.40}$$

While it is also possible to calculate the second term directly, we transform it in the following way:

$$-\int_0^1 dz \int_0^{1-z} dy \, \frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(x,y,z)\Big|_{x=1-z-y} = -2\int_0^1 dv \int_0^{1-v} du \, \mathcal{F}_V\,(1-v-u,u,v) . \tag{5.41}$$

A derivation of (5.41) is given in appendix A. Since

$$\begin{aligned}\mathcal{F}_V(1-v-u,u,v) &= -\frac{4}{\pi}\sin\left[\,\pi(1-v-u)\,\right]\,\sin\pi u\,\sin\pi v \\ &= -\frac{1}{\pi}\Big[\sin\left[\,2\pi(1-v-u)\,\right] + \sin 2\pi u + \sin 2\pi v\Big] \\ &= \frac{1}{\pi}\Big[\sin\left[\,2\pi(u+v)\,\right] - \sin 2\pi u - \sin 2\pi v\Big] , \end{aligned} \tag{5.42}$$

it is straightforward to calculate the integral:

$$\begin{aligned}& -2\int_0^1 dv \int_0^{1-v} du \, \mathcal{F}_V\,(1-v-u,u,v) \\ &= -\frac{2}{\pi}\int_0^1 dv \int_0^{1-v} du \Big[\sin\left[\,2\pi(u+v)\,\right] - \sin 2\pi u - \sin 2\pi v\Big] = \frac{3}{\pi^2} . \end{aligned} \tag{5.43}$$

The quantity $\mathcal{V}_3$ is therefore given by

$$\mathcal{V}_3 = \mathcal{F}_V(1,1,1) + \frac{3}{\pi^2} = \frac{3}{\pi^2} - \frac{3\sqrt{3}}{2\,\pi} . \tag{5.44}$$

6. Conclusions

We found that the equation of motion contracted with the solution itself (1.15) is not satisfied without the ψ_N piece in (1.5) because

$$\mathcal{K}_2 = \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{N}\langle\,\psi'_n, Q_B\psi'_m\,\rangle = -\frac{3}{\pi^2} + \mathcal{F}_K(1,1) , \tag{6.1}$$

$$\mathcal{V}_3 = \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{N}\sum_{k=0}^{N}\langle\,\psi'_n, \psi'_m * \psi'_k\,\rangle = \frac{3}{\pi^2} + \mathcal{F}_V(1,1,1) , \tag{6.2}$$

and $\mathcal{F}_K(1,1)+\mathcal{F}_V(1,1,1)\neq 0$, but it is satisfied when the ψ_N piece is included. The term $\mathcal{F}_K(1,1)$ in $\mathcal{K}_2$ and the term $\mathcal{F}_V(1,1,1)$ in $\mathcal{V}_3$ are canceled when the ψ_N piece is added, and the values (1.3) and (1.4) predicted by Sen's conjecture are nontrivially reproduced by the following integrals:

$$\langle\,\Psi, Q_B\Psi\,\rangle \;=\; -\int_0^1 dt\,\mathcal{F}_K\,(1-t,t) = -\,\frac{3}{\pi^2}\,, \tag{6.3}$$

$$\langle\,\Psi, \Psi*\Psi\,\rangle \;=\; -2\int_0^1 dv\int_0^{1-v} du\,\mathcal{F}_V\,(1-v-u,u,v) = \frac{3}{\pi^2}\,. \tag{6.4}$$

We emphasize that the cancellation between (6.3) and (6.4) is not a consequence of (1.6). We in fact do not have any deep understanding of why these apparently unrelated integrals should cancel, but it is really necessary for Schnabl's solution (1.5) to work out. It would be interesting, for example, if we could understand Schnabl's solution better in the context of noncommutative geometry underlying Witten's open cubic string field theory [1].

Acknowledgments

I would like to thank Ian Ellwood, Yoichi Kazama, Martin Schnabl, Wati Taylor, and Barton Zwiebach for useful discussions. I would also like to express my gratitude to Barton Zwiebach for his helpful comments and suggestions on an earlier version of the manuscript. This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FC02-94ER40818.

A. Integral

We derive (5.41) in this appendix. Let us first write

$$\begin{aligned} &-\int_0^1 dz\int_0^{1-z} dy\,\frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathcal{F}_V\,(x,y,z)\Big|_{x=1-z-y} \\ &= -\int_0^1 dv\int_0^{1-v} du\,\partial_y\partial_z\mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v}\,. \end{aligned} \tag{A.1}$$

Using the relation

$$[\,x\,\partial_x + y\,\partial_y + z\,\partial_z + 1\,]\,\partial_z\mathcal{F}_V(x,y,z) = 0\,, \tag{A.2}$$

which follows from (5.32), and

$$\partial_u\partial_z\mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} = (-\partial_x+\partial_y)\,\partial_z\mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v}\,, \tag{A.3}$$

$$\partial_v\partial_z\mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} = (-\partial_x+\partial_z)\,\partial_z\mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v}\,, \tag{A.4}$$

we obtain

$$\partial_y\partial_z\mathcal{F}_V(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v}$$

$$= [\,(1-u)\,\partial_u - v\,\partial_v - 1\,]\,\partial_z \mathcal{F}_V(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v}$$

$$= [\,\partial_u\,(1-u) - \partial_v\,v + 1\,]\,\partial_z \mathcal{F}_V(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} . \tag{A.5}$$

Note that

$$-\int_0^1 dv \int_0^{1-v} du\,\partial_u \left[(1-u)\,\partial_z \mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} \right] = 0\,, \tag{A.6}$$

and

$$\int_0^1 dv \int_0^{1-v} du\,\partial_v \left[v\,\partial_z \mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} \right]$$
$$= \int_0^1 du \int_0^{1-u} dv\,\partial_v \left[v\,\partial_z \mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} \right] = 0\,, \tag{A.7}$$

where we used that $\partial_z \mathcal{F}_V\,(0,y,z) = 0$ and $\partial_z \mathcal{F}_V\,(x,0,z) = 0$. We therefore obtain

$$-\int_0^1 dv \int_0^{1-v} du\,\partial_y \partial_z \mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v}$$
$$= -\int_0^1 dv \int_0^{1-v} du\,\partial_z \mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} . \tag{A.8}$$

Similarly, using the relation

$$[\,x\,\partial_x + y\,\partial_y + z\,\partial_z\,]\,\mathcal{F}_V(x,y,z) = 0\,, \tag{A.9}$$

which follows from (5.32), and

$$\partial_u \mathcal{F}_V\,(1-v-u,u,v) \;=\; (-\partial_x + \partial_y)\,\mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} , \tag{A.10}$$

$$\partial_v \mathcal{F}_V\,(1-v-u,u,v) \;=\; (-\partial_x + \partial_z)\,\mathcal{F}_V\,(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} , \tag{A.11}$$

we obtain

$$\partial_z \mathcal{F}_V(x,y,z)\Big|_{x=1-v-u,\,y=u,\,z=v} = [\,-u\,\partial_u + (1-v)\,\partial_v\,]\,\mathcal{F}_V(1-v-u,u,v)$$
$$= [\,-\partial_u u + \partial_v (1-v) + 2]\mathcal{F}_V(1-v-u,u,v). \tag{A.12}$$

Note that

$$\int_0^1 dv \int_0^{1-v} du\,\partial_u \Big[u\,\mathcal{F}_V\,(1-v-u,u,v) \Big] = 0\,, \tag{A.13}$$

and

$$-\int_0^1 dv \int_0^{1-v} du\,\partial_v \Big[(1-v)\,\mathcal{F}_V\,(1-v-u,u,v) \Big]$$
$$= -\int_0^1 du \int_0^{1-u} dv\,\partial_v \Big[(1-v)\,\mathcal{F}_V\,(1-v-u,u,v) \Big] = 0\,, \tag{A.14}$$

where we used that $\mathcal{F}_V(0,y,z)=0$ and $\mathcal{F}_V(x,y,0)=0$. We have therefore derived that

$$\begin{aligned}
&-\int_0^1 dv \int_0^{1-v} du\, \partial_y\partial_z \mathcal{F}_V(x,y,z)\Big|_{x=1-v-u,\, y=u,\, z=v} \\
&= -\int_0^1 dv \int_0^{1-v} du\, \partial_z \mathcal{F}_V(x,y,z)\Big|_{x=1-v-u,\, y=u,\, z=v} \\
&= -2\int_0^1 dv \int_0^{1-v} du\, \mathcal{F}_V(1-v-u,u,v)\ . \qquad \text{(A.15)}
\end{aligned}$$

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